# EXPLICIT EXAMPLES OF EQUIVALENCE RELATIONS AND II₁ FACTORS WITH PRESCRIBED FUNDAMENTAL GROUP AND OUTER AUTOMORPHISM GROUP

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ABSTRACT. In this paper we give a number of explicit constructions for $\mathrm{II}_1$ factors and $\mathrm{II}_1$ equivalence relations that have prescribed fundamental group and outer automorphism group. We construct factors and relations that have uncountable fundamental group different from $\mathbb{R}_+^*$. In fact, given any $\mathrm{II}_1$ equivalence relation, we construct a $\mathrm{II}_1$ factor with the same fundamental group.

Given any locally compact unimodular second countable group $G$, our construction gives a $\mathrm{II}_1$ equivalence relation $\mathcal{R}$ whose outer automorphism group is $G$. The same construction does not give a $\mathrm{II}_1$ factor with $G$ as outer automorphism group, but when $G$ is a compact group or if $G = \mathrm{SL}_n^{\pm}\mathbb{R} = \{g \in \mathrm{GL}_n\mathbb{R} \mid \det(g) = \pm 1\}$, then we still find a type $\mathrm{II}_1$ factor whose outer automorphism group is $G$.

## CONTENTS

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## Introduction

Type $II_1$ factors can be constructed in many different ways. For example as group von Neumann algebras or using the group-measure space construction. Two von Neumann algebras that are constructed in different ways can nevertheless be isomorphic. For example, all the group von Neumann algebras of ICC amenable groups are isomorphic [C1]. So a number of invariants for type $II_1$ factors were introduced. Probably the two most natural among them are the fundamental group and the outer automorphism group. Unfortunately, both are very hard to compute. For type $II_1$ equivalence relations, there are similar notions of fundamental group and outer automorphism group.

Our knowledge about fundamental groups and outer automorphism groups of type $II_1$ factors and equivalence relations advanced a lot in recent years. The recent development of Popa's deformation/rigidity theory allows us to study questions like "which groups appear as the fundamental group (or outer automorphism group) of a $II_1$ factor (or equivalence relation)." When we study this question, we will only be interested in separable $II_1$ factors, i.e. $II_1$ factors that are represented on a separable Hilbert space.

In this article I introduce a general method to create explicit examples of type $II_1$ factors and equivalence relations with prescribed fundamental group or prescribed outer automorphism group. This general method provides explicit examples in cases where we only knew pure existence results. For example, I give explicit examples of type $II_1$ factors with uncountable fundamental group different from $\mathbb{R}_+^*$. Existence of such factors was already proven in [PV2, PV5]. Moreover, I give an explicit construction for type $II_1$ factors with prescribed second countable compact outer automorphism group.

But I also give essentially new examples. I show that every second countable unimodular locally compact group is the outer automorphism group of a $II_1$ equivalence relation. This result does not extend immediately to $II_1$ factors, but I show that the groups $\mathrm{SL}_n^\pm \mathbb{R} = \{g \in \mathrm{GL}_n\mathbb{R} \mid \det(g) = \pm 1\}$ are outer automorphism groups of type $II_1$ factors. Using a different style of example, I show that every subgroup of $\mathbb{R}_+^*$ that appears as the fundamental group of a type $II_1$ equivalence relation also appears as the fundamental group of a $II_1$ factor.

The fundamental group was introduced by Murray and von Neumann in [MvN]. The fundamental group of a type $II_1$ factor is not related to the fundamental group of a topological space. Instead it is a group $\mathcal{F}(M) \subset \mathbb{R}_+^*$ of positive real numbers. Murray and von Neumann could compute the fundamental group in only one case. They showed that the fundamental group of the hyperfinite type $II_1$ factor $R$ is $\mathbb{R}_+^*$ itself. They raised the question: "What subgroups $\mathcal{F} \subset \mathbb{R}_+^*$ appear as the fundamental group of a separable type $II_1$ factor?"

The outer automorphism group $\mathrm{Out}(M)$ is the quotient group of all automorphisms of $M$ by the group of inner automorphisms, i.e. by the group $\mathrm{Inn}(M) = \{\mathrm{Ad}_u \mid u \in \mathcal{U}(M)\}$. Blattner shows that the group of outer automorphisms of $R$ is huge: it contains an automorphic copy of any second countable locally compact group [B]. I study the question: "which groups appear as the outer automorphism group of a separable type $II_1$ factor?"

Similar notions of fundamental group and outer automorphism group are defined for type $II_1$ equivalence relations.

The first significant progress on the questions raised above was realized by Connes in [C2]. He showed that the fundamental group and the outer automorphism group of a property (T) factor are countable. Examples of property (T) factors are the group von Neumann algebras $L(\Gamma)$ of property (T) ICC groups $\Gamma$. These were the first examples of $II_1$ factors with small fundamental group and outer automorphism group. However, he could only show that the invariants are countable, and even today we do not know the fundamental group nor the outer automorphism group of any property (T) factor.

Popa's Deformation/rigidity theory allows, among many other things, to compute outer automorphism groups and fundamental groups of type $II_1$ factors and equivalence relations. As a first application to this theory, Popa showed that the group von Neumann algebra $L(SL_2\mathbb{Z} \ltimes \mathbb{Z}^2)$ has trivial trivial fundamental group [P1], i.e. it contains only 1. Later he showed that any countable subgroup of $\mathbb{R}_+^*$ appears as the fundamental group of some type $II_1$ factor. Alternative constructions for the same result were given in [IPP, H]. In [PV2, PV5], Popa and Vaes show that a large class of uncountable subgroups of $\mathbb{R}_+^*$ appear as the fundamental group of a type $II_1$ factor and of a $II_1$ equivalence relation. However, they use a Baire category result to construct the type $II_1$ factor. In this paper I provide an explicit construction. This construction has several additional advantages over the original construction. These advantages will be detailed later on.

Moreover, I show that whenever a subgroup of $\mathbb{R}_+^*$ is the fundamental group of a type $II_1$ equivalence relation, then it is also the fundamental group of a type $II_1$ factor, see theorem 4.1.

The first actual computation of an outer automorphism group was done by Ioana, Peterson and Popa [IPP]. They show that every second countable compact abelian group appears as the outer automorphism group of a type $II_1$ factor. Again, their construction uses a Baire category argument. Later on Falgières and Vaes could extend this result and show that every (not necessarily abelian) second countable compact group appears as the outer automorphism group of a type $II_1$ factor. Here I provide an explicit construction for the latter.

For type $II_1$ equivalence relations, I even show that every second countable unimodular locally compact group appears as the outer automorphism group of a type $II_1$ equivalence relation. I have not been able to extend this result to outer automorphism groups of type $II_1$ factors. However, it is known that several important subclasses of unimodular groups appear as the outer automorphism group of a type $II_1$ factor. Vaes showed in [V2] that all countable discrete groups appear. As mentioned before, all compact groups appear too. Moreover, I show that all the groups of the form $SL_n^{\pm}\mathbb{R} = \{x \in GL_n\mathbb{R} \mid \det(x) = \pm 1\}$ with $n \in \mathbb{N}$ appear as the outer automorphism group of a type $II_1$ factor.

## An overview of the construction

Essentially all the results mentioned above use the same construction. This construction is very similar to the construction from [PV2, PV5].

We start with two actions. Firstly, a "general" ergodic infinite measure preserving action $\Lambda \curvearrowright (Y, \nu)$. Secondly, a specific action $\Gamma \curvearrowright (X, \mu)$. For the moment we only assume that this action is free, ergodic and probability measure preserving. Suppose we also have a quotient morphism $\pi : \Gamma \to \Lambda$. Then we can define a free infinite measure preserving action $\Gamma \curvearrowright X \times Y$ by the formula $g(x, y) = (gx, \pi(g)y)$. If we assume that $\ker \pi$ acts ergodically on $X$, then this new action is also ergodic.

The orbit equivalence relation $\mathcal{R} = \mathcal{R}(\Gamma \curvearrowright X \times Y)$ is a type $\mathrm{II}_\infty$ equivalence relation, and the von Neumann algebra $M = \mathrm{L}^\infty(X \times Y) \rtimes \Gamma$ is a type $\mathrm{II}_\infty$ factor. Let $U \subset X \times Y$ be a subset with finite measure. Then the restricted equivalence relation $\mathcal{R}|_U$ is a type $\mathrm{II}_1$ equivalence relation and the following short exact sequence describes the outer automorphism group and fundamental group of $\mathcal{R}|_U$ in terms of the outer automorphism group of $\mathcal{R}$.

$$1 \longrightarrow \mathrm{Out}(\mathcal{R}|_U) \longrightarrow \mathrm{Out}(\mathcal{R}) \xrightarrow{\mathrm{mod}} \mathcal{F}(\mathcal{R}|_U) \longrightarrow 1$$

Every orbit equivalence $\Delta$ of $\Gamma \curvearrowright X \times Y$ with itself scales the measure $\mu \times \nu$ by a constant $\mathrm{mod}(\Delta)$. Once we know $\mathrm{Out}(\mathcal{R})$ and the group morphism $\mathrm{mod} : \mathrm{Out}(\mathcal{R}) \to \mathbb{R}_+^*$, we also know the outer automorphism group and the fundamental group of $\mathcal{R}|_U$. The same relation holds between the outer automorphism group of the $\mathrm{II}_\infty$ factor $M$ and the outer automorphism group and fundamental group of a finite corner $pMp$.

Let $\Delta \in \mathrm{Aut}_{\mathrm{ns}}(Y, \nu)$ be a non-singular automorphism that commutes with the action of $\Lambda$. Then it is clear that $\mathrm{id} \times \Delta$ is an orbit equivalence of $\Gamma \curvearrowright X \times Y$. Because $\Gamma$ acts freely on $X$, such an orbit equivalence can never be inner. We denote by $\mathrm{Centr}_{\mathrm{Aut}_{\mathrm{ns}}(Y,\nu)}(\Lambda)$ the group of all non-singular automorphisms of $Y$ that commute with the action of $\Lambda$. So the construction above embeds $\mathrm{Centr}_{\mathrm{Aut}_{\mathrm{ns}}(Y,\nu)}(\Lambda)$ into $\mathrm{Out}(\mathcal{R})$.

Under strong conditions on the "specific" action $\Gamma \curvearrowright X$, this one-to-one group morphism is actually an isomorphism (i.e. it is also onto). In [PV2, PV5], Popa and Vaes give such a set of conditions on $\Gamma \curvearrowright X$. However, they could not give any explicit example of an action $\Gamma \curvearrowright X$ that satisfies their conditions. They used a Baire category argument to show the existence of such actions. Theorem A below gives an alternative set of conditions. This time, it is possible to give an explicit example of an action $\Gamma \curvearrowright X$ that satisfies the conditions.

**Theorem A** (see also theorem 2.1). *Let $\pi : \Gamma \to \Lambda$ be a quotient morphism and let $\Lambda \curvearrowright (Y, \nu)$ be an ergodic infinite measure preserving action. Suppose that the free, ergodic, probability measure preserving action $\Gamma \curvearrowright (X, \mu)$ satisfies the following conditions.*

- *The action $\ker \pi \curvearrowright X$ is $\mathcal{U}_{\mathrm{fin}}$ cocycle superrigid and weakly mixing. See subsection 1.2 for more details on cocycle superrigidity and weak mixing.*
- *There are no non-trivial group morphisms $\theta : \ker \pi \to \Lambda$.*
- *All the conjugations $\Delta$ of $\Gamma \curvearrowright X$ with itself are given by the formula $\Delta(x) = gx$ for some fixed element $g \in \Gamma$.*

*Consider the action $\Gamma \curvearrowright X \times Y$ given by $g(x, y) = (gx, \pi(g)y)$. Then the outer automorphism group of the type $II_\infty$ relation $\mathcal{R} = \mathcal{R}(\Gamma \curvearrowright X \times Y)$ is given by*

$$\mathrm{Out}(\mathcal{R}) = \mathrm{Centr}_{\mathrm{Aut}_{ns}(Y,\nu)}(\Lambda).$$

*In particular, for every subset $U \subset X \times Y$ with finite measure, we know that*

$$\mathcal{F}(\mathcal{R}|_U) = \mathrm{mod}(\mathrm{Centr}_{\mathrm{Aut}_{ns}(Y,\nu)}(\Lambda)) \ and \ \mathrm{Out}(\mathcal{R}|_U) = \mathrm{Centr}_{\mathrm{Aut}_{mp}(Y,\nu)}(\Lambda).$$

*Here we denoted the group of all measure preserving automorphisms of $Y$ by $\mathrm{Aut}_{mp}(Y, \nu)$.*

In fact, there is a relatively easy example of an action $\Gamma \curvearrowright X$ that satisfies the conditions of theorem A. We give a short description here, and more details can be found in proposition 2.3. Consider the action of $\Gamma = \mathrm{GL}_n\mathbb{Q} \ltimes \mathbb{Q}^n$ on $I = \mathbb{Q}^n$ by affine transformations. Define $\Gamma \curvearrowright (X, \mu) = (X_0, \mu_0)^I$ to be the generalized Bernoulli action over a purely atomic base space $(X_0, \mu_0)$ with unequal weights. Let $\pi : \Gamma \to \Lambda = \mathbb{Q}^\times$ be the determinant morphism. Then the action $\Gamma \curvearrowright (X, \mu)$ and the quotient $\pi : \Gamma \to \Lambda$ satisfy the conditions of theorem A.

In particular, for every infinite measure preserving ergodic action $\mathbb{Q}^\times \curvearrowright (Y, \nu)$, there is a type $II_1$ equivalence relation $\mathcal{R}$ with

$$\mathcal{F}(\mathcal{R}) = \mathrm{mod}(\mathrm{Centr}_{\mathrm{Aut}_{ns}(Y,\nu)}(\mathbb{Q}^\times)) \ and \ \mathrm{Out}(\mathcal{R}) = \mathrm{Centr}_{\mathrm{Aut}_{mp}(Y,\nu)}(\mathbb{Q}^\times).$$

Observe that we do not need the action $\mathbb{Q}^\times \curvearrowright Y$ to be free. Remark also that $\mathbb{Q}^\times$ is isomorphic to $\{\pm 1\} \times \mathbb{Z}^{(\mathbb{N})}$. So for every countable abelian group $\Lambda_0$ there is a quotient morphism $\mathbb{Q}^\times \to \Lambda_0$. So we can compose any action of $\Lambda_0$ with this quotient and obtain an action of $\mathbb{Q}^\times$.

In [A, AN], Aaronson and Nadkarni show that $\mathrm{mod}(\mathrm{Centr}_{\mathrm{Aut}_{ns}(Y,\nu)}(\mathbb{Z})) \subset \mathbb{R}_+^*$ can have any Hausdorff dimension between 0 and 1. This gives an explicit construction of type $II_1$ equivalence relations whose fundamental group can have any Hausdorff dimension. The existence of such equivalence relations was already shown in [PV2, PV5].

Let $\mathcal{G} \subset \mathrm{Aut}_{mp}(Y, \nu)$ be a closed abelian subgroup of measure preserving automorphisms on $(Y, \nu)$. Assume that $\mathcal{G}$ acts ergodically on $Y$. Because $\mathrm{Aut}_{mp}(Y, \nu)$ is a Polish group, we find a group morphism $\sigma : \mathbb{Q}^\times \to \mathcal{G}$ with dense range. This defines an ergodic measure preserving action of $\mathbb{Q}^\times$ on $Y$. Observe that the centralizer of this action is exactly $\mathrm{Centr}_{\mathrm{Aut}_{ns}(Y,\nu)}(\mathbb{Q}) = \mathrm{Centr}_{\mathrm{Aut}_{ns}(Y,\nu)}(\mathcal{G})$. From this it follows immediately that every second countable locally compact abelian group appears as the outer automorphism group of some type $II_1$ equivalence relation.

In the example above, the quotient group $\Lambda = \mathbb{Q}^\times$ is abelian. In section 7, we give a more complex example of an action $\Gamma \curvearrowright (X, \mu)$. But this time the corresponding quotient group $\Lambda$ is the free group $\mathbb{F}_\infty$. This has the following interesting consequence.

**Theorem B** (see also theorems 8.1 and 8.3)**.** *Let $\mathcal{G} \subset \mathrm{Aut}_{mp}(Y, \nu)$ be a closed group of measure preserving automorphisms of an infinite measure space, acting ergodically. Then there is a type $II_1$ equivalence relation $\mathcal{R}$ with*

$$\mathcal{F}(\mathcal{R}) = \mathrm{mod}(\mathrm{Centr}_{\mathrm{Aut}_{ns}(Y,\nu)}(\mathcal{G})) \ and \ \mathrm{Out}(\mathcal{R}) = \mathrm{Centr}_{\mathrm{Aut}_{mp}(Y,\nu)}(\mathcal{G}).$$

It follows that every second countable locally compact unimodular group appears as the outer automorphism group of some type $\text{II}_1$ equivalence relation.

The situation for type $\text{II}_1$ factors is very similar, but it requires an extra step. Remember that we want to control the outer automorphism group of $M = \text{L}^\infty(X \times Y) \rtimes \Gamma$. In general, let $\Gamma$ act freely on an infinite measure space $(Y, \nu)$. Denote $M = \text{L}^\infty(Y) \rtimes \Gamma$ and $\mathcal{R} = \mathcal{R}(\Gamma \curvearrowright Y)$. We have that

(1)                        $\text{Out}(\mathcal{R}) \ltimes \text{H}_1(\Gamma \curvearrowright Y) \subset \text{Out}(M).$

This inclusion can be strict, see [CJ, P3]. But if we know that every automorphism of $M$ preserves the Cartan subalgebra $\text{L}^\infty(Y)$ up to a unitary, then the inequality in (1) is actually an equality. It is a hard problem to decide if every automorphism of $M$ preserves $\text{L}^\infty(Y)$ but by now we know many classes of actions $\Gamma \curvearrowright Y$ with this property [IPP, P1, P2, P2, PV1, PV3].

Combining theorem A with the results from [IPP] yields the following result.

**Theorem C** (see theorem 5.1). *Let* $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ *be an amalgamated free product with* $\Gamma_1 \neq \Sigma \neq \Gamma_2$. *Let* $\pi : \Gamma \to \Lambda$ *be a quotient morphism. Assume that*

- $\Gamma_1 \cap \ker \pi$ *contains an infinite property (T) group* $G$.
- *the group* $\Sigma$ *is amenable.*
- *there are group elements* $g_1, \ldots, g_n \in \Gamma$ *such that the intersection* $\bigcap_{i=1}^n g_i \Sigma g_i^{-1}$ *is finite.*

*Let* $\Gamma \curvearrowright (X, \mu)$ *be a free, ergodic, p.m.p. action that satisfies the conditions of theorem A. For an arbitrary ergodic infinite measure preserving action* $\Lambda \curvearrowright (Y, \nu)$, *we define an action* $\Gamma \curvearrowright X \times Y$ *by the formula* $g(x, y) = (gx, \pi(g)y)$.

*Denote* $M = \text{L}^\infty(X \times Y) \rtimes \Gamma$ *and* $\mathcal{R} = \mathcal{R}(\Gamma \curvearrowright X \times Y)$. *Then we have that*

$$\text{Out}(M) = \text{Out}(\mathcal{R}) \ltimes \text{H}_1(\Gamma \curvearrowright X \times Y) = \text{Centr}_{\text{Aut}_{ns}(Y, \nu)}(\Lambda) \ltimes \text{H}_1(\Lambda \curvearrowright Y).$$

*In particular, for every finite projection* $p \in M$ *we find that*

$$\mathcal{F}(pMp) = \text{mod}(\text{Centr}_{\text{Aut}_{ns}(Y, \nu)}(\Lambda)) \ and \ \text{Out}(M) = \text{Centr}_{\text{Aut}_{mp}(Y, \nu)}(\Lambda) \ltimes \text{H}_1(\Lambda \curvearrowright Y).$$

Also in this case, we can give an explicit example of an action $\Gamma \curvearrowright (Y, \nu)$ that satisfies the conditions of theorem C. Just as in the equivalence relation case, we can choose the quotient group $\Lambda$ to be $\mathbb{F}_\infty$. For the fundamental group, we obtain the following.

**Theorem D.** *Let* $\mathcal{G} \subset \text{Aut}_{mp}(Y, \nu)$ *be a closed ergodic subgroup of (possibly infinite) measure preserving automorphisms. Then there is a type* $\text{II}_1$ *factor* $M_{\mathcal{G}}$ *with*

$$\mathcal{F}(M_{\mathcal{G}}) = \text{mod}(\text{Centr}_{\text{Aut}_{ns}(Y, \nu)}(\mathcal{G}))$$

For the outer automorphism group, things are a little more subtle. The cohomology group of an $\mathbb{F}_\infty$-action is usually a non-locally compact abelian group. Since we are most interested in locally compact outer automorphism groups, we want to make this cohomology group vanish, using a group different from $\mathbb{F}_\infty$. We succeeded in some specific cases.

**Theorem E** (see theorems 8.5 and 8.4).

- *If $\mathcal{G} \subset \mathrm{Aut}_{mp}(Y, \nu)$ is a closed (not necessarily free or ergodic) subgroup of* probability *measure preserving transformations, then we construct a type II$_1$ factor $M_{\mathcal{G}}$ with outer automorphism group*

$$\mathrm{Out}(M_{\mathcal{G}}) = \mathrm{Centr}_{\mathrm{Aut}_{mp}(Y,\nu)}(\mathcal{G}).$$

- *For every $n \in \mathbb{N}$, there is a type II$_1$ factor whose outer automorphism group is $\mathrm{SL}_n^{\pm}\mathbb{R}$.*

The first of these two statements gives an explicit construction for type II$_1$ factors with prescribed compact second countable outer automorphism group. But many more groups can be realized this way. For example $\mathrm{Aut}_{mp}(Y, \nu)$ for the non-atomic probability space $(Y, \nu)$ also appears as the outer automorphism group of a type II$_1$ factor.

In fact, we can further generalize theorem C, using some kind of first quantisation step. We replace the infinite measure space $(Y, \nu)$ by an arbitrary semifinite von Neumann algebra $(B, \mathrm{Tr})$ with a fixed semifinite trace. The automorphism group of $(Y, \nu)$ is replaced by the *outer* automorphism group of $B$. As before, the most general statement contains a 1-cohomology group of the action of $\Lambda$ on $B$. This group adds more technical problems than in the case of a measure space. I do not want to delve into these technicalities here. Therefore we only state the result for the fundamental group. More details are given in section 6. We obtain the following result.

**Theorem F** (see also theorem 8.2)**.** *Let $(B, \mathrm{Tr})$ be any properly infinite but semifinite von Neumann algebra with a specified semifinite trace. Let $\Lambda \subset \mathrm{Out}_{tp}(B, \mathrm{Tr})$ be a group of $\mathrm{Tr}$-preserving outer automorphisms of $B$. Assume that this group acts ergodically on the center of $B$. Then there is a type II$_1$ factor $M_{\Lambda}$ with fundamental group*

$$\mathcal{F}(M_{\Lambda}) = \mathrm{mod}(\mathrm{Centr}_{\mathrm{Out}(B)}(\Lambda)).$$

Unlike the other results in this paper, this is a pure existence result. We use Ozawa's result [O2] that for every $B$, there exists a property (T) group $G$ such that $G$ does not embed into the unitary group of $pBp$ for any finite projection $p \in B$. In many interesting cases however, we know an explicit example of such a group $G$. For example, with $B = \mathcal{B}(\ell^2(\mathbb{N})) \otimes R$ or $B = \mathcal{B}(\ell^2(\mathbb{N})) \otimes \mathrm{L}(\mathbb{F}_{\infty})$, we can take any property (T) group $G$.

Every fundamental group $\mathcal{F}(M)$ of a type II$_1$ factor appears in the way of theorem F:

$$\mathcal{F}(M) = \mathrm{mod}(\mathrm{Centr}_{\mathrm{Out}(\mathcal{B}(\ell^2(\mathbb{N})) ) \otimes M}(\{\mathrm{id}\})).$$

This gives an alternative characterisation of the set of all fundamental groups of type II$_1$ factors. But this characterization does not solve Murray and von Neumann's question, because it is not an intrinsic description. We conjecture that every fundamental group of a type II$_1$ factor appears as $\mathrm{mod}(\mathrm{Centr}_{\mathrm{Out}(\mathcal{B}(\ell^2(\mathbb{N}))) \otimes \mathrm{L}(\mathbb{F}_{\infty})}(\Lambda))$ for some subgroup $\Lambda$. If this conjecture would be true, then this would provide an important step in the solution of Murray and von Neumann's question.

## 1. Preliminaries and notations

### 1.1. The fundamental group and the outer automorphism group of $\text{II}_1$ equivalence relations and $\text{II}_1$ factors.

Murray and von Neumann defined [MvN] the fundamental group of a $\text{II}_1$ factor as

$$\mathcal{F}(M) = \{\tau(p)/\tau(q) \mid pMp \cong qMq\},$$

where $p$ and $q$ run over the nonzero projections in $M$. Even though this definition makes sense for all $\text{II}_1$ factors, we consider only *separable* von Neumann algebras (i.e. von Neumann algebras acting on a separable Hilbert space). There is a similar notion of fundamental group $\mathcal{F}(\mathcal{R})$ for a $\text{II}_1$ equivalence relation $\mathcal{R}$. A $\text{II}_1$ equivalence relation is an ergodic, countable, probability measure preserving equivalence relation on a standard probability space $(X, \mu)$. The fundamental group of $\mathcal{R}$ is defined by

$$\mathcal{F}(\mathcal{R}) = \{\mu(U)/\mu(V) \mid \mathcal{R}|_U \cong \mathcal{R}|_V\}$$

where $\mathcal{R}|_U$ denotes the restriction of $\mathcal{R}$ to $U$, i.e. for two points $x, y \in U$ we have that $x\mathcal{R}|_U y$ if and only if $x\mathcal{R}y$. Associated to a $\text{II}_1$ equivalence relation $\mathcal{R}$ is a separable $\text{II}_1$ factor $\text{L}(\mathcal{R})$ and a Cartan subalgebra $\text{L}^\infty(X, \mu) \subset \text{L}(\mathcal{R})$ via the generalized group-measure space construction of Feldman-Moore [FM]. The fundamental group of $\mathcal{R}$ is always a subgroup of $\mathcal{F}(\text{L}(\mathcal{R}))$. This inclusion can be strict: for example in [P5], based on results in [CJ], a $\text{II}_1$ equivalence relation $\mathcal{R}$ is constructed such that $\mathcal{F}(\mathcal{R}) = \{1\}$ while $\mathcal{F}(\text{L}(\mathcal{R})) = \mathbb{R}_+$. On the other hand, if for every $p, q \in A$ and every isomorphism $\alpha : pMp \to qMq$, the Cartan subalgebra $pA$ is mapped onto $uqAu^*$ for some unitary $u \in qMq$, then we have $\mathcal{F}(\text{L}(\mathcal{R})) = \mathcal{F}(\mathcal{R})$.

To a non-singular action $\Gamma \curvearrowright (X, \mu)$, one associates the orbit equivalence relation $\mathcal{R}(\Gamma \curvearrowright X)$ given by $x \sim y$ iff. $x = gy$ for some $g \in \Gamma$. If the action is ergodic and p.m.p. (probability measure preserving) on a standard probability space, then $\mathcal{R}(\Gamma \curvearrowright X)$ is a $\text{II}_1$ equivalence relation. If the action $\Gamma \curvearrowright X$ is free, then we have that $\text{L}(\mathcal{R}(\Gamma \curvearrowright X)) = \text{L}^\infty(X) \rtimes \Gamma$.

We will encounter ergodic actions $\Gamma \curvearrowright (X, \mu)$ preserving the infinite non-atomic standard measure $\mu$. The associated orbit equivalence relation $\mathcal{R}(\Gamma \curvearrowright X)$ is a so-called $\text{II}_\infty$ equivalence relation. For any $\text{II}_\infty$ equivalence relation $\mathcal{R}$, the restriction $\mathcal{R}|_U$ to a subset with $0 < \mu(U) < \infty$ is a $\text{II}_1$ equivalence relation. From now on, we only consider $\text{II}_1$ and $\text{II}_\infty$ equivalence relations.

To an equivalence relation $\mathcal{R}$ on $(X, \mu)$, several groups are associated. The full group $[\mathcal{R}]$ is the group of non-singular automorphisms $\Delta$ of $(X, \mu)$ with $\Delta(x) \sim x$ for almost all $x \in X$. Remark that all such $\Delta$ are measure preserving. An automorphism of $\mathcal{R}$ is a non-singular automorphism of $(X, \mu)$ such that $\Delta(x) \sim \Delta(y)$ iff $x \sim y$. We identify automorphisms that are equal almost everywhere. The group of all automorphisms of $\mathcal{R}$ will be denoted by $\text{Aut}(\mathcal{R})$. Note that such an automorphism $\Delta$ preserves the measure in the case of a $\text{II}_1$ relation, and scales the measure by a positive constant $\text{mod}(\Delta)$ if $\mathcal{R}$ is a $\text{II}_\infty$ relation. The full group $[\mathcal{R}]$ is a normal subgroup of $\text{Aut}(\mathcal{R})$. The quotient $\text{Out}(\mathcal{R}) = \text{Aut}(\mathcal{R})/[\mathcal{R}]$ is called the outer automorphism group. If $\mathcal{R}$ is a $\text{II}_\infty$ relation and $0 < \mu(U) < \infty$, then we have the following short exact sequence

$$1 \longrightarrow \text{Out}(\mathcal{R}|_U) \longrightarrow \text{Out}(\mathcal{R}) \xrightarrow{\text{mod}} \mathcal{F}(\mathcal{R}|_U) \longrightarrow 1$$

In particular, if we can identify $\text{Out}(\mathcal{R})$, we also know $\mathcal{F}(\mathcal{R}|_U)$ and $\text{Out}(\mathcal{R}|_U)$.

The situation is similar for factors. Let $M$ be a factor. We denote by $\text{Aut}(M)$ the group of all automorphisms of $M$. If $M$ is a type $\text{II}_1$ factor, then every automorphism of $M$ preserves the trace, and if $M$ is of type $\text{II}_\infty$, then every automorphism scales the trace by a positive constant $\text{mod}(\psi) \in \mathbb{R}_+^*$. An automorphism $\psi : M \to M$ is called inner if it is given by unitary conjugation, i.e. there is a unitary $u \in M$ such that $\psi(x) = uxu^*$ for all $x \in M$. The group of all inner automorphisms is denoted by $\text{Inn}(M)$. This group is a normal subgroup of $\text{Aut}(M)$, and we call the quotient $\text{Out}(M) = \text{Aut}(M)/\text{Inn}(M)$ the outer automorphism group of $M$.

If $M$ is a type $\text{II}_\infty$ factor and $p \in M$ is a projection with finite trace, then we have the short exact sequence

$$1 \longrightarrow \text{Out}(pMp) \longrightarrow \text{Out}(M) \xrightarrow{\text{mod}} \mathcal{F}(pMp) \longrightarrow 1$$

### 1.2. Cocycle superrigidity and $\mathcal{U}_{\mathbf{fin}}$ groups.

Let $\Gamma \curvearrowright (X, \mu)$ be a probability measure preserving action. We say that $\Gamma \curvearrowright (X, \mu)$ is weakly mixing if the diagonal action of $\Gamma$ on $X \times X$ is ergodic. Many equivalent definitions are known, see for example [V1, Appendix D]. For us, the most important equivalent definition is the following.

**Lemma 1.1** (see [PV4, lemma 5.4]). *Let $\Gamma \curvearrowright (X, \mu)$ be a weakly mixing action.*

*Let $(Z, d)$ be a Polish space with separable complete metric $d$. Suppose that a Polish group $\Lambda$ has a continuous action $\alpha$ by isometries on $Z$.*

*Let $\Gamma \curvearrowright (Y, \nu)$ be another measure preserving action on a standard measure space and let $\omega : \Gamma \times Y \to \Lambda$ be a measurable map. If a measurable map $f : X \times Y \to Z$ satisfies*

$$f(gx, gy) = \alpha_{\omega(g,y)} f(x, y),$$

*then there is a measurable map $f_0 : Y \to Z$ such that $f(x, y) = f_0(y)$ almost everywhere.*

For example, all (plain) Bernoulli actions of infinite groups are weakly mixing. More generally, a generalized Bernoulli action $\Gamma \curvearrowright (X_0, \mu_0)^I$ is weakly mixing iff. it is ergodic iff. every orbit of $\Gamma \curvearrowright I$ is infinite.

Fix a Polish group $\Lambda$. A measurable map $\omega : \Gamma \times X \to \Lambda$ is called a 1-cocycle if $\omega$ satisfies the cocycle relation

$$\omega(g, hx)\omega(h, x) = \omega(gh, x) \text{ for almost every } x \in X \text{ and } g, h \in \Gamma.$$

Examples are given by Zimmer cocycles: Let $\Lambda \curvearrowright (Y, \nu)$ be a free non-singular action and let $\Delta : X \to Y$ be an orbit equivalence (i.e. $\Delta$ is an isomorphism between the orbit equivalence relations $\mathcal{R}(\Gamma \curvearrowright X)$ and $\mathcal{R}(\Lambda \curvearrowright Y)$, or still $\Delta$ is a non-singular isomorphism such that $\Delta(\Gamma x) = \Lambda \Delta(x)$ for almost all $x \in X$). The Zimmer cocycle is the unique (up to measure 0) map $\omega : \Gamma \times X \to \Lambda$ such that $\Delta(gx) = \omega(g, x)\Delta(x)$ almost everywhere. If $\Delta_1, \Delta_2$ are two orbit equivalences and there is an $\eta \in [\mathcal{R}(\Lambda \curvearrowright Y)]$ with $\Delta_1 = \eta \circ \Delta_2$, then $\omega_1$ and $\omega_2$ are homologous. This means that there is a function $\varphi : X \to \Lambda$ such that

$$\omega_1(g, x) = \varphi(gx)^{-1}\omega_2(g, x)\varphi(x) \text{ for all } g \text{ and almost all } x$$

The group morphisms $\delta : \Gamma \to \Lambda$ can be identified with the cocycles that do not depend on the $X$-variable. Popa showed the following cocycle superrigidity theorem. To state it, we need the notion of a $\mathcal{U}_{\mathrm{fin}}$ group. A Polish group $\Lambda$ is called a $\mathcal{U}_{\mathrm{fin}}$ group if it can be realized as a closed subgroup of the unitary group of a finite von Neumann algebra. Examples include all countable groups and all compact second countable groups.

**Theorem 1.2** ([P4, theorem 0.1]). *Suppose that $\Gamma \curvearrowright (X, \mu)$ is an s-malleable action (for example a generalized Bernoulli action) of a group $\Gamma$ with an infinite almost normal subgroup $G$ that has the relative property (T). Assume that $G$ acts weakly mixingly on $(X, \mu)$. Then every cocycle $\omega : \Gamma \times X \to \Lambda$, with a $\mathcal{U}_{\mathrm{fin}}$ target group $\Lambda$, is cohomologous to a group morphism.*

Groups $\Gamma$ with infinite almost normal subgroups $G$ that have the relative property (T) are called weakly rigid. A subgroup $G \subset \Gamma$ is called almost normal if $G \cap gGg^{-1}$ has finite index in $G$, for all $g \in \Gamma$.

We say that an action $\Gamma \curvearrowright (X, \mu)$ is $\mathcal{U}_{\mathrm{fin}}$-cocycle superrigid if every cocycle $\omega : \Gamma \times X \to \Lambda$, to a $\mathcal{U}_{\mathrm{fin}}$ target group $\Lambda$, is cohomologous to a group morphism. For example, [P4, theorem 0.1] asserts that s-malleable actions of weakly rigid groups are $\mathcal{U}_{\mathrm{fin}}$-cocycle superrigid.

1.3. **Intertwining by bimodules in the semifinite case.** Given two subalgebras $A$ and $B$ in a finite von Neumann algebra $M$, we want to decide if $A$ and $B$ are unitarily conjugate or not. In [P2, section 2] and [P1, theorem A.1], Popa introduced a powerful technique to decide this type of question. In fact, we will need a similar technique that works if $M$ is a semifinite von Neumann algebra instead of a finite von Neumann algebra. Such a generalization was developed in [CH] and later an improved version was given in [HR]. We give an overview of the terminology and of the results that we will use.

Let $(B, \mathrm{Tr}_B)$ be a von Neumann algebra with a faithful normal semifinite trace. From now on, a trace will always be assumed to be faithful and normal. Any right Hilbert $B$-module $H$ is isomorphic to a submodule of $\ell^2(\mathbb{N}) \otimes \mathrm{L}^2(B, \mathrm{Tr}_B)$, so there is a projection $p \in \mathcal{B}(\ell^2(\mathbb{N})) \overline{\otimes} B$ such that $H \cong p(\ell^2(\mathbb{N}) \otimes \mathrm{L}^2(B, \mathrm{Tr}_B))$. The trace $\mathrm{Tr} \otimes \mathrm{Tr}_B(p)$ of $p$ is an invariant of the module $H$, which we call the dimension $\dim_{\mathrm{Tr}_B}(H_B)$. Observe that the dimension of $H$ depends on the choice of trace $\mathrm{Tr}$, and even in the case of a $\mathrm{II}_\infty$ *factor* $B$, the canonical trace $\mathrm{Tr}$ is only determined up to a positive scalar multiple.

**Definition/Theorem 1.3** ([HR, lemma 2.2], generalizing [P2, theorem 2.1]). *Let $(M, \mathrm{Tr})$ be a von Neumann algebra with a semifinite trace. Let $p \in M$ be a projection with finite trace and let $A \subset pMp$ be a von Neumann subalgebra. Suppose that $B \subset M$ is a unital von Neumann subalgebra such that the restricted trace $\mathrm{Tr}\,|_B$ is still semifinite. We say that $A$ embeds into $B$ over $M$ (denoted by $A \prec_M B$) if one of the following equivalent conditions holds.*

- *There is an $A$-$B$ subbimodule $H \subset p\mathrm{L}^2(M)$ that has finite dimension over $(B, \mathrm{Tr})$.*
- *There exist a natural number $n$, a projection $q \in \mathrm{M}_n(\mathbb{C}) \otimes B$ with finite trace, a partial isometry $v \in p(\mathrm{M}_{1,n}(\mathbb{C}) \otimes M)q$ and a unital $*$-homomorphism $\theta : A \to q(\mathrm{M}_n(\mathbb{C}) \otimes B)q$ such that $av = v\theta(a)$ for all $a \in A$.*
- *There is no sequence $(v_n)_n$ of unitaries in $A$ such that*
$$\|\mathrm{E}_B(xv_ny)\|_{\mathrm{Tr},2} \to 0 \text{ for all } x, y \in M.$$

*In terms of Popa's notion for finite von Neumann algebras, this is equivalent with*

- *There is a projection $q \in B$ with finite trace such that $A \prec_{(p \vee q)M(p \vee q)} qBq$.*

Let $C \subset M$ be a unital subalgebra. If there is a unitary $u \in M$ with $uCu^* \subset B$, then we have $pCp \prec_M B$ for every projection $p \in A$ with finite trace. The converse implication does not hold in general, but if both $C$ and $B$ are Cartan subalgebras, then the fact that $pC \prec_M B$, for some $p \in C$ with finite trace, is sufficient to conclude that $C$ is unitarily conjugate to $B$. In the case where $M$ is a $\mathrm{II}_1$ factor, this fact was proven in [P1, theorem A.1]. For a proof of the generalization to the semifinite case, see [HR, proposition 2.3]. In proposition 3.2, we give a different set of conditions on two subalgebras $B, C \subset M$ such that $pBp \prec_M C$ and $qCq \prec_M B$ implies that $B$ and $C$ are unitarily conjugate in $M$.

Suppose that $M = B \rtimes \Gamma$ where $(B, \mathrm{Tr})$ is a von Neumann algebra with a semifinite trace and $\Gamma$ acts on $B$ preserving $\mathrm{Tr}$. Let $A \subset pMp$ be a regular subalgebra of a finite corner of $M$. Assume that we have $A \prec_M B \rtimes \Sigma$ for some (highly non-normal) subgroup $\Sigma \subset \Gamma$. In [PV1, theorem 6.16], a criterion is given to show that then we also have $A \prec_M B$. An improved version of this criterion is given in [HPV]: For any subgroup $\Sigma \subset \Gamma$, we denote by $\mathrm{H}_\Sigma$ the closed linear span of all $A$-$B \rtimes \Sigma$ subbimodules of $p\mathrm{L}^2(M)$ that have finite dimension over $B \rtimes \Sigma$. Because $A$ is regular, this space $\mathrm{H}_\Sigma$ is invariant under left multiplication by $pMp$, so $\mathrm{H}_\Sigma$ is of the form $\mathrm{H}_\Sigma = p\mathrm{L}^2(M)z_\Sigma$ for some projection $z_\Sigma$ in the relative commutant of $B \rtimes \Sigma$ inside $M$. This projection $z_\Sigma$ is uniquely determined if we also require that $z_\Sigma$ is smaller than the central support of $p$. By definition, we have $z_\Sigma \neq 0$ if and only if $A \prec_M B \rtimes \Sigma$.

Given $g \in \Gamma$, we have $z_{g\Sigma g^{-1}} = u_g z_\Sigma u_g^*$. For two subgroups $\Sigma_1, \Sigma_2 \subset \Gamma$, [HPV] shows that $z_{\Sigma_1 \cap \Sigma_2} = z_{\Sigma_1} z_{\Sigma_2}$. Remember that $z_\Sigma$ is always contained in $M \cap (B \rtimes \Sigma)'$. Suppose that $u_g$ commutes with all of $M \cap (B \rtimes \Sigma)'$ for some $g \in \Gamma$. Then the combination of both observations in the beginning of this paragraph shows that we have $z_{\Sigma \cap g\Sigma g^{-1}} = z_\Sigma$. By induction, they obtain the following.

**Corollary 1.4** (see [HPV, Corollary 7]). *Let $M = B \rtimes \Gamma$ be a the crossed product of an action of $\Gamma$ preserving a semifinite trace $\mathrm{Tr}$ on $B$. Suppose that a regular subalgebra $A \subset pMp$ of a finite corner of $M$ satisfies $A \prec_M B \rtimes \Sigma$, for a subgroup $\Sigma \subset \Gamma$. If there are $g_1, \ldots, g_n \in \Gamma$ such that*

- *the $u_{g_i}$ are contained in the relative bicommutant of $B \rtimes \Sigma$, i.e.*

$$u_{g_i} \in M \cap \big( M \cap (B \rtimes \Sigma)' \big)',$$

- *the intersection $\bigcap_{i=1}^n g_i \Sigma g_i^{-1}$ is finite,*

*then we also have $A \prec_M B$.*

Actually, [HPV, Corollary 7] is stated with a finite trace $\mathrm{Tr}$, but its proof is still valid in the semifinite case. A second result in [HPV] states that if $M$ is a $\mathrm{II}_1$ factor and if $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ is an amalgamated free product with $\Gamma_1 \neq \Sigma \neq \Gamma_2$, then $z_\Sigma$ is either 1 or 0. The proof uses the finiteness of the trace. However, the following variant is still true. Although the proof is almost identical to the proof of [HPV, Proposition 8], we provide a proof for the convenience of the reader.

**Proposition 1.5** (see [HPV, Proposition 8]). *Let $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ be an amalgamated free product with $\Gamma_1 \neq \Sigma \neq \Gamma_2$. Let $\pi : \Gamma \to \Lambda$ be a quotient morphism such that $\ker \pi$ is not contained in $\Sigma$. Let $\Gamma \curvearrowright (X, \mu)$ be a free, p.m.p. action, and let $\Lambda \curvearrowright (B, \mathrm{Tr})$ be any trace preserving action on a semifinite von Neumann algebra. Consider the action of $\Gamma$ on $\mathrm{L}^\infty(X) \otimes B$ given by $g(a \otimes b) = ga \otimes \pi(g)b$, and denote the crossed product by $M = (\mathrm{L}^\infty(X) \otimes B) \rtimes \Gamma$.*

*Let $A \subset pMp$ be a regular subalgebra of a finite corner of $M$. Then the projections $z_\Sigma$, $z_{\Gamma_1}$ and $z_{\Gamma_2}$, as defined above, coincide.*

*In particular, this projection $z_\Sigma = z_{\Gamma_1} = z_{\Gamma_2}$ is contained in the center of $M$. So whenever $A \prec_M (\mathrm{L}^\infty(X) \otimes B) \rtimes \Gamma_i$ for $i = 1$ or $i = 2$, then it follows that*

$$A \underset{M}{\prec} (\mathrm{L}^\infty(X) \otimes B) \rtimes (g_1 \Sigma g_1^{-1} \cap \ldots \cap g_n \Sigma g_n^{-1})$$

*for any finite number of elements $g_1, \ldots, g_n \in \Gamma$.*

*Proof.* We show that $z_\Sigma = z_{\Gamma_1}$. By symmetry it then follows that $z_\Sigma = z_{\Gamma_2}$. Observe that $z_\Sigma$ and $z_{\Gamma_1}$ are contained in the abelian von Neumann algebra $\mathrm{L}^\infty(X) \otimes \mathcal{Z} B$. Moreover, $z_\Sigma$ is smaller than $z_{\Gamma_2}$.

Denote by $S$ the set of all elements $g \in \Gamma$ that have a reduced expression whose first letter comes from $\Gamma_2 - \Sigma$. For all $g \in S$, we see that $g\Gamma_1 g^{-1} \cap \Gamma_1 \subset \Sigma$, so it follows that $z_{\Gamma_1} u_g z_{\Gamma_1} u_g^* \leq z_\Sigma$. We claim that

$$q = \bigvee_{g \in S} z_{\Gamma_1} u_g z_{\Gamma_1} u_g^* = z_{\Gamma_1}.$$

Suppose this was not the case, then we denote $r = z_{\Gamma_1} - q$. Whenever $g \in S$, it is clear that

$$r \wedge u_g r u_g^* \leq u_g z_{\Gamma_1} u_g^* \wedge z_{\Gamma_1} \wedge r = 0$$

.

On the other hand, there are nonzero projections in the abelian von Neumann algebras $r_1 \in \mathrm{L}^\infty(X)$ and $r_2 \in \mathcal{Z}(B)$ such that $r_1 \otimes r_2 \leq r$. In particular, $r_1$ is orthogonal to $u_g r_1 u_g^*$ for all $g \in \ker \pi \cap S$. If we can show that $\ker \pi \cap S$ is infinite, we would have shown that the probability space $X$ contains infinitely many mutually disjoint subsets of equal, nonzero measure. This contradiction proves that $r = z_{\Gamma_1}$ and hence that $z_\Sigma = z_{\Gamma_1}$, finishing the proof of proposition 1.5.

We still have to show that $\ker \pi \cap S$ is infinite. In fact, we show that there is an element $g \in \ker \pi$ of infinite order and such that $g^n \in S$ for all $n \in \mathbb{N}$. Let $g \in \ker \pi \setminus \Sigma$ be any element. We consider four cases. If the first letter of $g$ comes from $\Gamma_2 - \Sigma$ and its last letter comes from $\Gamma_1 - \Sigma$, then $g^n$ is clearly contained in $S$ for all $n \in \mathbb{N}$. If the first letter comes from $\Gamma_1 - \Sigma$ and the last letter comes from $\Gamma_2 - \Sigma$, then we have $g^{-n} \in S$. If both the first and the last letter of $g$ come from $\Gamma_1 - \Sigma$, take any $h \in \Gamma_2 \setminus \Sigma$. Then $hgh^{-1}g$ is an element of $\ker \pi$ whose first letter comes from $\Gamma_2 - \Sigma$ and whose last letter comes from $\Gamma_1 - \Sigma$. Finally, if both the first and the last letter of $g$ come from $\Gamma_2 \setminus \Sigma$, then we take any element $h \in \Gamma_1 - \Sigma$, and we see that $(ghgh^{-1})^n \in S$ for all $n \in \mathbb{N}$.  $\square$

## 2. The equivalence relation case

In this section we concentrate on fundamental groups and outer automorphism groups of type $\mathrm{II}_1$ equivalence relations. We mainly prove theorem A, but in section 6, we will meet cases where the action $\Lambda \curvearrowright (Y, \nu)$ does not preserve any $\sigma$-finite measure on $Y$. At the end of this section, in proposition 2.3, we give an explicit example of an action $\Gamma \curvearrowright (X, \mu)$ that satisfies the conditions of theorem 2.1. This gives explicit examples of type $\mathrm{II}_1$ equivalence relations with uncountable fundamental group different from $\mathbb{R}_+^*$.

**Theorem 2.1** (see also theorem A). *Let $\Gamma \curvearrowright (X, \mu)$ be a free, ergodic, p.m.p. action and let $\pi : \Gamma \to \Lambda$ be a quotient morphism. Suppose that*

- *the action of $\ker \pi$ on $X$ is weakly mixing and $\mathcal{U}_{\mathrm{fin}}$-cocycle superrigid.*
- *there are no non-trivial group morphisms $\theta : \ker \pi \to \Lambda$.*
- $\mathrm{Norm}_{\mathrm{Aut}_{mp}(X, \mu)}(\ker \pi) = \Gamma$, *i.e. all the conjugations of the action of $\ker \pi$ on $X$ are described by elements of $\Gamma$.*

*Let $\Lambda \curvearrowright (Y, \nu)$ be any ergodic non-singular action on a standard measure space, and consider the action $\Gamma \curvearrowright X \times Y$ given by $g(x, y) = (gx, \pi(g)y)$. Then the orbit equivalence relation $\mathcal{R} = \mathcal{R}(\Gamma \curvearrowright X \times Y)$ has outer automorphism group*

$$\mathrm{Out}(\mathcal{R}) = \mathrm{Centr}_{\mathrm{Aut}_{ns}(Y)}(\Lambda).$$

*If the action $\Lambda \curvearrowright (Y, \nu)$ preserves the infinite measure $\nu$, we recover theorem A:*

$$\mathcal{F}(\mathcal{R}|_U) = \mathrm{mod}(\mathrm{Centr}_{\mathrm{Aut}_{ns}(Y)}(\Lambda)) \ and \ \mathrm{Out}(\mathcal{R}|_U) = \mathrm{Centr}_{\mathrm{Aut}_{mp}(Y, \nu)}(\Lambda),$$

*for any subset $U \subset Y$ with finite measure.*

In the proof of theorem 2.1, we encounter local automorphisms of a measure space $(X, \mu)$. We say that a map $\Delta : X \to X$ is a local automorphism if there is a countable partition $X = \bigsqcup_i X_i$ into Borel subsets, such that the restriction $\Delta|_{X_i}$ is a non-singular isomorphism between the Borel sets $X_i$ and $\Delta(X_i)$.

*Proof.* The formula $\Delta_0 \mapsto \mathrm{id}_X \times \Delta_0$ defines an injective group morphism from the centralizer $\mathrm{Centr}_{\mathrm{Aut}_{ns}(Y)}(\Lambda)$ to the outer automorphism group $\mathrm{Out}(\mathcal{R})$.

It remains to show that every orbit equivalence $\Delta : X \times Y \to X \times Y$ is of the form $\mathrm{id}_X \times \Delta_0$, up to an inner automorphism. Let $\Delta$ be such an orbit equivalence and consider its Zimmer cocycle $\omega : \Gamma \times X \times Y \to \Gamma$. We can consider the restriction $\omega|_{\ker \pi}$ as a cocycle $\tilde{\omega} : \ker \pi \times X \to \mathcal{G}$ to the $\mathcal{U}_{\mathrm{fin}}$ target group $\mathcal{G}$ of measurable functions from $Y$ to $\Gamma$. Cocycle superrigidity yields a measurable function $\varphi : X \times Y \to \Gamma$ and group morphisms $\delta_y : \ker \pi \to \Gamma$ such that

$$\delta_y(g) = \varphi(gx, y)\omega(g, x, y)\varphi(x, y)^{-1}$$

for all $g \in \ker \pi$ and for almost all $(x, y) \in X \times Y$. Because there are no non-trivial group morphisms $\theta : \ker \pi \to \Lambda$, it follows that $\delta_y(\ker \pi) \subset \ker \pi$. Then [P4, proposition 3.6] says that $\varphi(g(x, y))\omega(g, x, y)\varphi(x, y)$ is essentially independent of $x$, for all $g \in \Gamma$. We denote that unique essential value by $\delta(g, y)$.

We define a local automorphism $\widetilde{\Delta} : X \times Y \to X \times Y$ by the formula $\widetilde{\Delta}(x,y) = \varphi(x,y)\Delta(x,y)$. Observe that $\widetilde{\Delta}(g(x,y)) = \delta_y(g)\widetilde{\Delta}(x,y)$ for all $g \in \ker \pi$ and for almost all $(x,y) \in X \times Y$. Split $\widetilde{\Delta}$ into its components: $\widetilde{\Delta} = (\Delta_1(x,y), \Delta_2(x,y))$. Remember that $\delta_y(\ker \pi)$ is contained in $\ker \pi$. The function $\Delta_2$ satisfies

$$\Delta_2(gx,y) = \Delta_2(x,y) \text{ for all } g \in \ker \pi \text{ and for almost all } (x,y) \in X \times Y.$$

Since $\ker \pi$ acts ergodically on $X$, we see that $\Delta_2(x,y)$ does not depend on $x$. We write just $\Delta_2(y)$.

*Claim: we show that the formula $\Delta_{1,y}(x) = \Delta_1(x,y)$ defines and automorphism $\Delta_{1,y} : X \to X$, for almost every $y \in Y$.* We can apply the same argument as before to $\Theta = \Delta^{-1}$. So we find a local automorphism $\widetilde{\Theta} : X \times Y \to X \times Y$ and group morphisms $\theta_y : \ker \pi \to \ker \pi$ such that $\widetilde{\Theta}(gx,y) = \theta_y(g)\widetilde{\Theta}(x,y)$ for all $g \in \ker \pi$ and almost all $(x,y) \in X \times Y$. Moreover, we can assume that the second component $\Theta_2$ of $\widetilde{\Theta}$ does not depend on the $X$-variable, and we consider it as $\Theta_2 : Y \to Y$. We denote the first component of $\widetilde{\Theta}$ by $\Theta_1$, and we write $\Theta_{1,y}(x) = \Theta_1(x,y)$. So we have that $\widetilde{\Theta}(x,y) = (\Theta_{1,y}(x), \Theta_2(y))$ almost everywhere.

Remark that $\widetilde{\Delta} \circ \widetilde{\Theta}$ and $\widetilde{\Theta} \circ \widetilde{\Delta}$ are inner local automorphisms in the sense that there are measurable maps $\varphi, \psi : X \times Y \to \Gamma$ such that

$$\widetilde{\Delta}\left(\widetilde{\Theta}\left(x,y\right)\right) = \varphi(x,y)(x,y) \qquad \widetilde{\Theta}\left(\widetilde{\Delta}\left(x,y\right)\right) = \psi(x,y)(x,y) \qquad \text{almost everywhere}$$

It follows that

$$\varphi(gx,y)g(x,y) = \widetilde{\Delta}\left(\widetilde{\Theta}\left(gx,y\right)\right) = \delta_{\Theta_2(y)}(\theta_y(g))\varphi(x,y)(x,y)$$

for all $g \in \ker \pi$ and almost all $(x,y) \in X \times Y$. By freeness, we see that $\varphi(gx,y) = \delta_{\Theta_2(y)}(\theta_y(g))\varphi(x,y)g^{-1}$ almost everywhere. Weak mixing (see for example [PV4, lemma 5.4]) implies that $\varphi(x,y)$ does not depend on $x$, so we write $\varphi(y) = \varphi(x,y)$ a.e. The same argument works for $\psi$, and we also write just $\psi(y)$.

Then it follows that $\Delta_2 \circ \Theta_2$ and $\Theta_2 \circ \Delta_2$ are inner and hence local automorphisms. Lemma 2.2 shows that $\Delta_2, \Theta_2$ and the $\Delta_{1,y}, \Theta_{1,y}$ are local automorphisms, for almost all $y \in Y$. Now we see that $\psi(y)^{-1} \circ \Theta_{1,\Delta_2(y)}$ is a left inverse for $\Delta_{1,y}$, for almost every $y \in Y$. Similarly, $\Theta_{1,y} \circ \varphi(y)^{-1}$ is a right inverse for $\Delta_{1,\Theta_2(y)}$. So for almost all $y$ in the non-negligible set $W = \Theta_2(Y)$, we know that $\Delta_{1,y}$ is a genuine automorphism of $X$. But for any $g \in \Gamma$, we see that $\delta(g,y) \circ \Delta_{1,y} = \Delta_{1,\pi(g)y} \circ g$. The ergodicity of $\Lambda \curvearrowright Y$ shows that in fact almost every $\Delta_{1,y}$ is a genuine automorphism. This finishes the proof of our claim.

Observe that $\Delta_{1,y}$ is a conjugation for the action $\ker \pi \curvearrowright X$, for almost every $y \in Y$. Our third condition shows that $\Delta_{1,y} \in \Gamma$, so there is an element $\psi(y) \in \Gamma$ such that $\Delta_{1,y}(x) = \psi(y)^{-1}x$ almost everywhere. Set $\Delta_0(y) = \psi(y)^{-1}\Delta_2(y)$ and observe that $\psi(y)^{-1}\widetilde{\Delta}(x,y) = (x, \Delta_0(y))$

almost everywhere. Hence, for every $g \in \Gamma$, we see that

$$
\begin{aligned}
&(gx, \Delta_0(\pi(g)y)) \\
&= \psi(\pi(g)y)^{-1}\varphi(g(x,y))\Delta(g(x,y)) \\
&= \underbrace{\psi(\pi(g)y)^{-1}\varphi(g(x,y))\omega(g,x,y)\varphi(x,y)^{-1}\psi(y)}_{\overset{\|}{\widetilde{\omega}(g,x,y)}}(x, \Delta_0(y))
\end{aligned}
$$

Since $\Gamma$ acts freely on $X$, we see that $\widetilde{\omega}(g,x,y) = g$ almost everywhere, and in particular, $\Delta_0$ commutes with the action of $\Lambda$. We still have to show that $\Delta_0$ is an automorphism of $Y$. The same argument as before applies to $\Delta^{-1}$, so we find a local automorphism $\Theta_0 : Y \to Y$ such that $(x, \Theta_0(\Delta_0(y))) \in \Gamma(x,y)$ almost everywhere. By freeness we see that $\Theta_0 \circ \Delta_0 = \mathrm{id}_Y$. Symmetrically, we find that $\Delta_0 \circ \Theta_0 = \mathrm{id}_Y$, so $\Delta_0$ is an automorphism of $Y$.

We have shown that $\Delta$ is of the form $\mathrm{id}_X \times \Delta_0$ up to an inner automorphism. $\qquad \square$

**Lemma 2.2.** *Suppose that* $\Delta \colon X \times Y \to X \times Y$ *and* $\Theta \colon X \times Y \to X \times Y$ *are local automorphisms. Suppose that the second component of* $\Delta$ *and* $\Theta$ *does not depend on the* $X$*-variable, so we can write*

$$\Delta(x,y) = (\Delta_{1,y}(x), \Delta_2(y)) \ \text{ and } \ \Theta(x,y) = (\Theta_{1,y}(x), \Theta_2(y))$$

*almost everywhere.*

*If* $\Delta_2 \circ \Theta_2$ *and* $\Theta_2 \circ \Delta_2$ *are local automorphisms of* $Y$*, then* $\Delta_2$*,* $\Theta_2$*, the* $\Delta_{1,y}$ *and the* $\Theta_{1,y}$ *for almost all* $y \in Y$ *are local automorphisms.*

*Proof.* We first show that $\Delta_2$ and $\Theta_2$ are local automorphisms.
Remark that $\Delta^{-1}(X \times W) = X \times \Delta_2^{-1}(W)$ for all measurable subsets $W \subset Y$. Since $\Delta$ is non-singular, we see that $\Delta_2$ is non-singular. The same argument applies to $\Theta_2$.

We show that $\Delta_2$ and $\Theta_2$ are partial automorphisms. We can partition $Y$ into a countable disjoint union of subsets $Y = \bigsqcup_i Y_i$ such that every $\Psi_i = \Theta_2 \circ \Delta_2|_{Y_i}$ is a non-singular automorphism. Similarly, we can partition $Y$ into a disjoint union $Y = \bigsqcup_j \widetilde{Y}_j$ such that every $\Phi_j = \Delta_2 \circ \Theta_2|_{\widetilde{Y}_j}$ is a non-singular automorphism. For every $i, j$, it follows that $\Delta_2|_{Y_i \cap \Delta_2^{-1}(\widetilde{Y}_j)}$ is a non-singular automorphism between $Y_i \cap \Delta_2^{-1}$ and $\widetilde{Y}_j \cap \Theta_2^{-1}(\Psi_i(Y_i \cap \Delta_2^{-1}(\widetilde{Y}_j)))$, so $\Delta_2$ is a local automorphism. The same argument applies to $\Theta_2$.

*Then we show that almost all the* $\Delta_{1,y}$ *are local automorphisms.*
Let $W \subset Y$ be a subset such that $\Delta_2|_W$ is a non-singular isomorphism onto its image. Let $U$ be a measurable subset of $X \times W$ such that $\Delta|_U$ is a non-singular isomorphism onto its image $V = \Delta(U)$. We denote by $U_y$ the slice $U_y = \{x \in X \mid (x,y) \in U\}$. Consider the Radon-Nikodym derivative $F \colon V \to \mathbb{R}_+^*$ of $\Delta|_U$ and denote by $G \colon \Delta_2(Y_i) \to \mathbb{R}_+^*$ the Radon-Nikodym derivative of $\Delta_2|_{Y_i}$. Then we compute that, for all measurable functions $f : X \to \mathbb{R}$

and $g : Y \to \mathbb{R}$, we have

$$\int_{Y_i} g(\Delta_2(y)) \int_{U_y} f(\Delta_{1,y}(x)) \, \mathrm{d} \, \mu(x) \, \mathrm{d} \, \nu(y)$$

$$= \int_U (f \otimes g)(\Delta(x,y)) \, \mathrm{d}(\mu \times \nu)(x,y)$$

$$= \int_V (f \otimes g)(x,y) F(x,y) \, \mathrm{d}(\mu \times \nu)(x,y)$$

$$= \int_{Y_i} g(\Delta_2(y)) \int_{V_{\Delta_2(y)}} f(x) \frac{F(x, \Delta_2(y))}{G(\Delta_2(y))} \, \mathrm{d} \, \mu(x) \, \mathrm{d} \, \nu(y).$$

This calculation shows that the Radon-Nikodym derivative of $\Delta_{1,y}|_{U_y}$ is given by the formula $H_y(x) = F(x, \Delta_2(y))/G(\Delta_2(y))$ and in particular, we see that $\Delta_{1,y}|_{U_y}$ is non-singular. Because $\Delta|_U$ is bijective, it follows that $\Delta_{1,y}|_{U_y}$ is a non-singular isomorphism onto its image, for almost all $y \in Y$ with $\mu(U_y) > 0$.

We can partition $X \times Y$ into a countable disjoint union of sets like $U$, so $\Delta_{1,y}$ is a local automorphism for almost all $y \in Y$.                                                                                $\square$

**Proposition 2.3.** *Consider the action of the affine group $\Gamma = \mathrm{GL}_n \mathbb{Q} \ltimes \mathbb{Q}^n$ on the countable affine space $I = \mathbb{Q}^n$, with dimension $n \geq 2$. The determinant gives us a quotient morphism $\pi : \Gamma \to \Lambda = \mathbb{Q}^{\times}$. Then the generalized Bernoulli action $\Gamma \curvearrowright (X, \mu) = (X_0, \mu_0)^I$, with atomic base space $(X_0, \mu_0)$ with unequal weights, satisfies the conditions of theorem A.*

Observe that $\mathbb{Q}^{\times} \cong \mathbb{Z}/2 \oplus \mathbb{Z}^{\oplus \infty}$. We do not need the action of $\mathbb{Q}^{\times}$ on $(Y, \nu)$ to be free. This way we see already that for any abelian group $\Lambda$ and any ergodic, measure preserving action of $\Lambda$ on $(Y, \nu)$, we can realize $\mathrm{mod}(\mathrm{Centr}_{\mathrm{Aut}_{\mathrm{ns}}(Y)}(\Lambda))$ as the fundamental group of a type $\mathrm{II}_1$ equivalence relation. In particular, we can do this with $\Lambda = \mathbb{Z}$. Using ergodic measures on $\mathbb{R}$, Aaronson and Nadkarni [A, AN] show that for any number $0 \leq \alpha \leq 1$, there is a measure preserving action $\mathbb{Z} \curvearrowright Y$ such that $\mathrm{mod}(\mathrm{Centr}_{\mathrm{Aut}_{\mathrm{ns}}(Y)}(\mathbb{Z}))$ has Hausdorff dimension $\alpha$.

*Proof.* We show that the action $\Gamma \curvearrowright (X, \mu)$ satisfies the conditions of theorem A. The second condition follows from the fact that $\mathrm{SL}_n \mathbb{Q}$ has no non-trivial morphisms to abelian groups and that the conjugates of $\mathrm{SL}_n \mathbb{Q}$ generate $\mathrm{SL}_n \mathbb{Q} \ltimes \mathbb{Q}^n$. For cocycle superrigidity, observe that $\mathbb{Z}^n$ is an almost normal subgroup of $\mathrm{SL}_n \mathbb{Q} \ltimes \mathbb{Q}^n$ that has relative property (T) and that acts with infinite orbits on $I$. Popa's cocycle superrigidity theorem [P4, theorem 0.1] implies that $\ker \pi \curvearrowright X$ is $\mathcal{U}_{\mathrm{fin}}$-cocycle superrigid.

Let $\Delta : X \to X$ be a conjugation of $\ker \pi \curvearrowright X$. We show that $\Delta \in \Gamma$. By [V2, proposition 6.10] there is a measure preserving automorphism $\Delta_0 : X_0 \to X_0$ and a conjugation $\alpha : I \to I$ such that $\Delta(x)_{\alpha(i)} = \Delta_0(x_i)$. Since $(X_0, \mu_0)$ is atomic with unequal weights, $\Delta_0$ can only be the identity. Moreover, there is an automorphism $\delta : \mathrm{SL}_n \mathbb{Q} \ltimes \mathbb{Q}^n \to \mathrm{SL}_n \mathbb{Q} \ltimes \mathbb{Q}^n$ such that $\alpha(gi) = \delta(g)\alpha(i)$ for all $i \in I$ and $g \in \mathrm{SL}_n \mathbb{Q} \ltimes \mathbb{Q}^n$. As every affine subspace $V \subset I = \mathbb{Q}^n$ is of the form $V = \mathrm{Fix}(g)$ for some $g \in \mathrm{SL}_n \mathbb{Q} \ltimes \mathbb{Q}^n$, the conjugation $\alpha$ maps affine subspaces of $\mathbb{Q}^n$ to affine subspaces. Since $\alpha$ preserves inclusions, it also preserves the dimension and in particular, $\alpha$ maps lines to lines and planes to planes. Such a map is easily seen to be an affine transformation of $\mathbb{Q}^n$, i.e. $\alpha \in \Gamma$. We obtained that $\Delta \in \Gamma$.                                            $\square$

## 3. A PRESERVATION OF CARTAN RESULT

Let $\Gamma \curvearrowright (X, \mu)$ be a free, ergodic, measure preserving action. For the orbit equivalence relation $\mathcal{R} = \mathcal{R}(\Gamma \curvearrowright X)$, the previous section allows us to compute the group $\mathrm{mod}(\mathrm{Aut}(\mathcal{R}))$, in specific cases. For the group measure space construction $M = \mathrm{L}^\infty(X) \rtimes \Gamma$, it is clear that $\mathrm{mod}(\mathrm{Aut}(M))$ contains at least $\mathrm{mod}(\mathrm{Aut}(\mathcal{R}))$. Equality holds if every automorphism $\psi : M \to M$ maps $\mathrm{L}^\infty(X)$ onto a unitary conjugate of $\mathrm{L}^\infty(X)$ inside $M$.

The following theorem provides a criterion to show, for some actions $\Gamma \curvearrowright X$, that every automorphism $\psi$ of $M = \mathrm{L}^\infty(X) \rtimes \Gamma$ preserves the Cartan subalgebra $\mathrm{L}^\infty(X)$ up to a unitary conjugation in $M$. Its proof is a direct combination of results from [IPP]. The statement is very similar to [PV3, theorem 5.2]. Because the exact form we need is not available in the literature, we provide a complete proof.

**Theorem 3.1** (see [IPP]). *Let $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ be an amalgamated free product and consider a quotient morphism $\pi : \Gamma \to \Lambda$. Let $(B, \mathrm{Tr})$ be a von Neumann algebra with a semifinite trace $\mathrm{Tr}$ and assume that $B$ does not have minimal projections. Let $\beta; \Lambda \curvearrowright B$ be a $\mathrm{Tr}$-preserving action. Assume that*

- *$\Gamma_1 \cap \ker \pi$ contains an infinite property (T) group $G$.*
- *the group $\Sigma$ is amenable and $\Gamma_1 \neq \Sigma \neq \Gamma_2$.*
- *there is no $*$-homomorphism $\theta : \mathrm{L}(G) \to qBq$ for any projection $q \in B$ with finite trace.*
- *there are group elements $g_1, \ldots, g_n \in \Gamma$ such that the intersection $\bigcap_{i=1}^n g_i \Sigma g_i^{-1}$ is finite.*

*Let $\alpha : \Gamma \curvearrowright (X, \mu)$ be any free, ergodic, p.m.p. action. Consider the action $\sigma : \Gamma \curvearrowright \mathrm{L}^\infty(X) \overline{\otimes} B$ given by $\sigma_g(a \otimes b) = \alpha_g(a) \otimes \beta_{\pi(g)}(b)$. Denote by $M$ the crossed product $M = (\mathrm{L}^\infty(X) \overline{\otimes} B) \rtimes \Gamma$.*

*Then any $*$-automorphism $\psi$ of $M$ stably preserves $C = \mathrm{L}^\infty(X) \overline{\otimes} B$ up to a unitary, i.e. there is a unitary $u \in \mathcal{B}(\ell^2(\mathbb{N})) \overline{\otimes} \mathcal{B}(\ell^2(\mathbb{N})) \overline{\otimes} M$ such that*

$$u(\ell^\infty(\mathbb{N}) \overline{\otimes} \mathcal{B}(\ell^2(\mathbb{N})) \overline{\otimes} \psi(C))u^* = \ell^\infty(\mathbb{N}) \overline{\otimes} \mathcal{B}(\ell^2(\mathbb{N})) \overline{\otimes} C$$

*If $B$ is abelian, or if $B$ is properly infinite and the ergodic action $\Lambda \curvearrowright \mathcal{Z}(B)$ does not preserve a probability measure (i.e. the orbit equivalence relation is of type $I_\infty$, $II_\infty$ or $III$), then there is a unitary $v \in M$ such that $v\psi(C)v^* = C$.*

*Proof.* Throughout the proof, we will use the intertwining-by-bimodules technique in the semifinite setting. Because [IPP] is formulated a finite setting, need to be able to take finite corners. The following claim allows to do exactly that.

Write $A = \mathrm{L}^\infty(X)$. Denote by $\mathcal{N} = B \rtimes \Lambda$ and set $\mathcal{M} = (\mathrm{L}^\infty(X) \rtimes \Gamma) \overline{\otimes} \mathcal{N}$. There is a natural trace preserving embedding $\varphi$ of $M$ into $\mathcal{M}$, given by the formula $\varphi((a \otimes b)u_g) = au_g \otimes bu_{\pi(g)}$. *Claim: let $H \subset \Gamma$ be a subgroup, let $A_0$ be $\mathbb{C}$ or $A$, and let $Q \subset pMp$ be a subalgebra of a finite corner of $M$, then we have that*

$$Q \underset{M}{\prec} (A_0 \otimes B) \rtimes H \text{ if and only if } \varphi(Q) \underset{\mathcal{M}}{\prec} (A_0 \rtimes H) \otimes \mathcal{N} .$$

If $Q$ embeds into $(A_0 \otimes B) \rtimes H$ inside $M$, it is clear that $\varphi(Q)$ embeds into $\varphi((A_0 \otimes B) \rtimes H) \subset (A_0 \rtimes H) \otimes \mathcal{N}$ inside $\varphi(M) \subset \mathcal{M}$.

Conversely, assume that $Q$ does not embed into $(A_0 \otimes B) \rtimes H$ inside $M$. Then, by [HR, theorem 2.1], we find a sequence $(v_n)_n$ of unitaries in $Q$ such that

$$\left\| \mathrm{E}_{(A_0 \otimes B) \rtimes H}(x v_n y) \right\|_2 \to 0 \text{ for all } x, y \in M.$$

We have to show that

$$\left\| \mathrm{E}_{(A_0 \rtimes H) \overline{\otimes} \mathcal{N}}(x \varphi(v_n) y) \right\|_2 \to 0 \text{ for all } x, y \in \mathcal{M}.$$

By Kaplanski's density theorem, it suffices to do this for $x = u_g$ and $y = u_h$. Write the Fourier expansion of $v_n$ in $(A_0 \otimes B) \rtimes \Gamma$ as $v_n = \sum_{k \in \Gamma} v_{n,k} u_k$. Then we compute that

$$\left\| \mathrm{E}_{(A_0 \rtimes H) \overline{\otimes} \mathcal{N}}(u_g \varphi(v_n) u_h) \right\|_2^2 = \sum_{\substack{k \in \Gamma, gkh \in H}} \| \mathrm{E}_{A_0 \otimes B}(v_{n,k}) \|_2^2$$

$$= \left\| \mathrm{E}_{(A_0 \otimes B) \rtimes H}(u_g v_n u_h) \right\|_2^2 \to 0,$$

proving that $\varphi(Q)$ does not embed into $(A_0 \rtimes H) \otimes \mathcal{N}$ inside $\mathcal{M}$.

Using this claim, we prove theorem 3.1. Let $\psi$ be an automorphism of $M$ and let $p \in B$ be a projection with finite trace. Up to a unitary conjugation, we can assume that $q = \psi(p)$ is contained in $B$. We will use the following notations

$$\widetilde{\mathcal{M}}_1 = (A \rtimes \Gamma_1) \overline{\otimes} q\mathcal{N}q$$
$$\widetilde{\mathcal{M}}_2 = (A \rtimes \Gamma_2) \overline{\otimes} q\mathcal{N}q$$
$$\widetilde{\mathcal{P}} = (A \rtimes \Sigma) \overline{\otimes} q\mathcal{N}q$$
$$\widetilde{\mathcal{M}} = (A \rtimes \Gamma) \overline{\otimes} q\mathcal{N}q = \widetilde{\mathcal{M}}_1 *_{\widetilde{\mathcal{P}}} \widetilde{\mathcal{M}}_2$$
$$\widetilde{\psi} = \varphi \circ \psi : pMp \to \widetilde{\mathcal{M}}$$

*Step 1: we show that $\widetilde{\psi}(A \otimes pBp) \prec_{\widetilde{\mathcal{M}}} \widetilde{\mathcal{M}}_i$ for $i = 1$ or 2* As in [IPP], we consider the word-length deformation $(m_\rho)_\rho$ on $\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}_1 *_{\widetilde{\mathcal{P}}} \widetilde{\mathcal{M}}_2$, i.e. for every number $0 < \rho < 1$, we define a unital completely positive map $m_\rho : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}$ that is given by $m_\rho(au_g \otimes x) = \rho^{|g|}(au_g \otimes x)$ for all $a \in A, g \in \Gamma$ and $x \in q\mathcal{N}q$. The notation $|g|$ denotes the word-length of $g$ in the amalgamated free product $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$. Observe that $\|m_\rho(x) - x\|_2$ tends to 0 as $\rho$ tends to 1, pointwise for all $x \in \widetilde{\mathcal{M}}$.

The subalgebra $Q = \widetilde{\psi}(p\,\mathrm{L}(G))$ has property (T), so the word-length deformation converges to id uniformly on the unit ball of $Q$. We also know that $\psi(p\,\mathrm{L}(G))$ does not embed into $B$ inside $M$, so the claim above shows that $Q$ does not embed into $q\mathcal{N}q$ inside $\widetilde{\mathcal{M}}$. Because $A \rtimes \Sigma$ is amenable, it follows that $Q$ does not embed into $\widetilde{\mathcal{P}}$ inside $\widetilde{\mathcal{M}}$ (see [P1]). The unitaries $\widetilde{\psi}(pu_g) \in \widetilde{\mathcal{M}}$ with $g \in G$, normalize the abelian von Neumann algebra $\widetilde{\psi}(pA) \subset \widetilde{\mathcal{M}}$. By [PV3, lemma 5.7] (which is a version of [IPP, proposition 1.4.1]), we find a $0 < \rho_0 < 1$ and a $\delta > 0$ such that $\mathrm{Tr}(w^* m_{\rho_0}(w)) \geq \delta$ for all unitaries $w \in \widetilde{\psi}(p(A \rtimes G))$.

We also know that $\widetilde{\psi}(p(A \rtimes G))$ does not embed into $\mathcal{P}$ inside $\mathcal{M}$, so [PV3, theorem 5.4] (which is a version of [IPP, theorem 4.3]) implies that $\mathcal{N}_{\widetilde{\mathcal{M}}}(\widetilde{\psi}(p(A \rtimes G)))''$ embeds into $\widetilde{\mathcal{M}}_i$, for $i = 1$ or $2$. This normalizer contains $\widetilde{\psi}(A \otimes pBp)$.

*We finish the proof of theorem 3.1.*
We have just shown that $\widetilde{\psi}(A \otimes pBp)$ embeds into $\widetilde{\mathcal{M}}_i$ inside $\widetilde{\mathcal{M}}$. The claim in the beginning of this proof shows that $\psi(A \otimes pBp)$ embeds into $M_1$ inside $M$. Because $\ker \pi$ contains the infinite property (T) group $G$, it is certainly not contained in the amenable group $\Sigma$. By our last condition, we can apply proposition 1.5 (which is a variant of [HPV, proposition 8]) to obtain that $\psi(A \otimes pBp) \prec_M A \otimes B$.

If $B$ happens to be abelian, then $A \otimes B$ is a Cartan subalgebra of $M$, so [HPV, proposition 2.3] implies that $\psi(A \otimes B)$ is unitarily conjugate to $A \otimes B$.

Otherwise, we have just shown that $\psi(A \otimes pBp) \prec_M A \otimes B$, and by symmetry it follows that $A \otimes qBq \prec_M \psi(A \otimes B)$. Proposition 3.2 below shows that $\psi(A \otimes B)$ is stably unitarily conjugate to $A \otimes B$ inside $M$, i.e. there is a unitary $u \in \mathcal{B}(\ell^2(\mathbb{N})) \otimes \mathcal{B}(\ell^2(\mathbb{N})) \otimes M$ such that

$$u(\ell^\infty(\mathbb{N}) \otimes \mathcal{B}(\ell^2(\mathbb{N})) \otimes \psi(A \otimes B))u^* = \ell^\infty(\mathbb{N}) \otimes \mathcal{B}(\ell^2(\mathbb{N})) \otimes A \otimes B.$$

If $B$ is properly infinite and there is no finite $\Lambda$-invariant measure on $\mathcal{Z}(B)$, proposition 3.2 below shows that $\psi(A \otimes B)$ is actually unitarily conjugate to $A \otimes B$.     $\square$

Popa's intertwining-by-bimodules technique is used to decide if two subalgebras $A, B \subset M$ of a $II_1$ factor are unitarily conjugate. If $A$ and $B$ are unitarily conjugate, the it is clear that $A \prec_M B$ and $B \prec_M A$. The converse implication is false in general. However, there are several special cases in which this converse implication holds.

For example, if $A$ and $B$ are Cartan subalgebras of $M$ and $A \prec_M B$, then [P1, theorem A.1] (or [HR, proposition 2.3] in the semifinite case) shows that $A$ and $B$ are unitarily conjugate. We need a criterion that allows non-abelian algebras $A$ and $B$. Such a criterion is given below.

**Proposition 3.2.** *Let $M$ be a type $II_\infty$ factor and let $A, B \subset M$ be von Neumann subalgebras such that the restricted traces $\mathrm{Tr}\,|_A$ and $\mathrm{Tr}\,|_B$ are still semifinite. Suppose that*

- *$A$ and $B$ are regular subalgebras of $M$.*
- *$\mathcal{Z}(A)' \cap M = A$ and $\mathcal{Z}(B)' \cap M = B$.*

*If $pAp \prec_M B$ and $qBq \prec_M A$ for some projection $p \in A$ and $q \in B$ with finite trace, then $A$ and $B$ are stably unitarily conjugate in $M$, in the sense that*

$$A \otimes \mathcal{B}(\ell^2(\mathbb{N})) \otimes \ell^\infty(\mathbb{N}) \quad and \quad B \otimes \mathcal{B}(\ell^2(\mathbb{N})) \otimes \ell^\infty(\mathbb{N})$$

*are unitarily conjugate subalgebras of $M \otimes \mathcal{B}(\ell^2(\mathbb{N})) \otimes \mathcal{B}(\ell^2(\mathbb{N}))$*

*If moreover $A$ and $B$ satisfy*

- *$A$ and $B$ are properly infinite, i.e. every central projection in $A$ respectively $B$ is an infinite projection in $A$ respectively $B$,*

- *there is no finite trace $\tau_{\mathcal{Z}(A)}$ on $\mathcal{Z}(A)$ that is invariant under the action of $\mathcal{N}_M(A)$. Similarly there is no $\mathcal{N}_M(B)$-invariant finite trace $\tau_{\mathcal{Z}(B)}$ on $\mathcal{Z}(B)$,*

*then there is a unitary $u \in M$ with $u^*Au = B$*

*Proof. Step 1: There are projections $p_1 \in A$ and $q_1 \in B$ such that $p_1 \mathrm{L}^2(M)q_1$ contains a nonzero finite index $p_1Ap_1$-$q_1Bq_1$ subbimodule.*
Consider the $A$-$B$ bimodules

$$\mathrm{H}_{A \prec_M B} = \bigvee \left\{ K \subset p_1 \mathrm{L}^2(M) \, \middle| \, \begin{array}{l} p_1 \in A \text{ is a projection with } \mathrm{Tr}(p_1) < \infty, \text{ and} \\ K \text{ is a } p_1Ap_1\text{-}B \text{ bimodule with } \dim_{\mathrm{Tr}}(K_B) < \infty \end{array} \right\}$$

$$\mathrm{H}_{A \succ_M B} = \bigvee \left\{ K \subset \mathrm{L}^2(M)q_1 \, \middle| \, \begin{array}{l} q \in B \text{ is a projection with } \mathrm{Tr}(q_1) < \infty, \text{ and} \\ K \text{ is an } A\text{-}q_1Bq_1 \text{ bimodule with } \dim_{\mathrm{Tr}}({}_A K) < \infty \end{array} \right\}$$

Because $A, B \subset M$ are regular subalgebras, both bimodules are actually non-zero $M$-$M$ bimodules, so it follows that $\mathrm{H}_{A \prec_M B} = \mathrm{L}^2(M) = \mathrm{H}_{A \succ_M B}$. In particular, we find that, for any projections $p_1 \in A$ and $q_1 \in B$ with finite trace, the Hilbert space $p_1 \mathrm{L}^2(M)q_1$ is spanned by $p_1Ap_1$-$q_1Bq_1$ bimodules that have finite dimension over $q_1Bq_1$. Similarly, $p_1 \mathrm{L}^2(M)q_1$ is spanned by $p_1Ap_1$-$q_1Bq_1$ bimodules that have finite dimension over $p_1Ap_1$. Write $C$ for the von Neumann algebra of $p_1Ap_1$-$q_1Bq_1$ bimodular operators on $p_1 \mathrm{L}^2(M)q_1$. On $C$, we have two natural traces: a trace $\mathrm{Tr}_A$ that comes from the canonical trace on the commutant of $p_1Ap_1$, and a trace $\mathrm{Tr}_B$ that comes from the canonical trace on the commutant of $q_1Bq_1$. The fact that $\mathrm{H}_{A \prec_M B} = \mathrm{L}^2(M) = \mathrm{H}_{A \succ_M B}$ implies that both traces $\mathrm{Tr}_A$ and $\mathrm{Tr}_B$ are semifinite traces on $C$. In particular, we find elements $x, y \in C$ such that $\mathrm{Tr}_A(x^*x) < \infty$, such that $\mathrm{Tr}_B(y^*y) < \infty$ and such that $xy \neq 0$. Then we also have that $\mathrm{Tr}_A(x^*y^*yx) < \infty$ and $\mathrm{Tr}_B(x^*y^*yx) < \infty$, so any nonzero spectral projection $r = \chi_{[\delta, \infty[}(y^*x^*xy)$ is the projection onto a nonzero finite index $p_1Ap_1$-$q_1Bq_1$ subbimodule $K$ of $p_1 \mathrm{L}^2(M)q_1$.

*Step 2: Write $A_1 = p_1Ap_1$ and $B_1 = q_1Bq_1$. We write the left action of $A_1$ on $K$ explicitly by $\lambda : A_1 \to \mathcal{B}(K)$ and we write the right action by $\rho : B_1^{op} \to \mathcal{B}(K)$. Then there is a non-zero $A_1$-$B_1$ bimodular projection $r$ on $K$ such that*

$$r\lambda(\mathcal{Z}(A_1)) = r({}_{A_1} \mathcal{B}_{B_1}(K))r = r\rho(\mathcal{Z}(B_1)).$$

Denote by $C = {}_{A_1} \mathcal{B}_{B_1}(K)$ the von Neumann algebra of all $A_1$-$B_1$ bimodular operators on $K$, so $C = \lambda(A_1)' \cap \rho(B_1^{\mathrm{op}})'$. Since $K$ is a finite index bimodule, we know that $\lambda(A_1) \subset \rho(B_1^{\mathrm{op}})'$ is a finite index inclusion, so $\lambda(\mathcal{Z}(A_1))$ is a finite index subalgebra of $C$, see for example [V2, lemma A.3]. Because $\mathcal{Z}(A_1)$ is an abelian von Neumann algebra, we find a nonzero projection $r_1 \in C$ that satisfies $r_0Cr_0 = \lambda(\mathcal{Z}(A_1))p$. Similarly, we see that $\rho(\mathcal{Z}(B_1))r_0$ is a finite index subalgebra of $r_0Cr_0$, so we find a nonzero projection $r \leq r_0$ in $C$ such that $rCr = \rho(\mathcal{Z}(B))r$.

*Step 3: There is a partial isometry $v \in M$ such that $vv^* \in A$ and $v^*v \in B$, both having finite trace, and such that $v^*Av = v^*vBv^*v$. Possibly passing to a subbimodule, we can assume that $\widetilde{K} = rK$ is finitely generated over $B_1$. So we find an $A_1$-$B_1$ bimodular unitary*

$$V : {}_{\psi(A_1)}q_2(\mathbb{C}^n \otimes \mathrm{L}^2(B_1))_{B_1} \to {}_{A_1} \widetilde{K}_{B_1}$$

for some projection $q_2 \in B_1^n$ and a finite index inclusion $\psi : A_1 \to q_2 B_1^n q_2$. The conclusion of step 2 shows that $\psi(\mathcal{Z}(A_1)) = \psi(A_1)' \cap q_2 B_1^n q_2 = \mathcal{Z}(q_2 B_1^n q_2)$.

Consider the vector $\xi = [V(e_1 \otimes \hat{1}), \dots, V(e_n \otimes \hat{1})] \in \mathbb{C}^n \otimes \mathrm{L}^2(M)$. Observe that this vector satisfies the relation $x\xi = \xi\psi(x)$ for all $x \in A_1$. Polar decomposition gives a partial isometry $v \in \mathrm{M}_{1,n} \otimes M$ such that $xv = v\psi(x)$ for all $x \in A_1$. Moreover, $vv^* \leq p_1$ has finite trace. Remark that $vv^*$ commutes with $A_1$, so $vv^*$ is contained in $A_1 \subset A$. Also, $v^*v$ commutes with $\psi(A_1)$. In particular, it commutes with $\psi(\mathcal{Z}(A_1)) = \mathcal{Z}(q_2 B_1^n q_2)$. Hence we have that $v^*v \in q_2 B_1^n q_2 \cap \psi(A_1)' = \mathcal{Z}(q_2 B_1^n q_2)$. Possibly making $p_1$, $q_1$ and $q_2$ smaller, we can assume that $n = 1$, $p_1 = vv^*$ and $v^*v = q_2 = q_1$. Now we have that $v^* \mathcal{Z}(p_1 A p_1)v = \mathcal{Z}(q_1 B q_1)$. Our first condition shows that $v^* A v = v^*vBv^*v$.

*Step 4: The subalgebras $A$ and $B$ are stably unitarily conjugate.* We use the following notations:

$$\begin{aligned} \mathcal{A} &= A \otimes \mathcal{B}(\ell^2(\mathbb{N})) & \widetilde{\mathcal{A}} &= \mathcal{A} \otimes \ell^\infty(\mathbb{N}) = A \otimes \mathcal{B}(\ell^2(\mathbb{N})) \otimes \ell^\infty(\mathbb{N}) \\ \mathcal{B} &= B \otimes \mathcal{B}(\ell^2(\mathbb{N})) & \widetilde{\mathcal{B}} &= \mathcal{B} \otimes \ell^\infty(\mathbb{N}) = B \otimes \mathcal{B}(\ell^2(\mathbb{N})) \otimes \ell^\infty(\mathbb{N}) \\ \mathcal{M} &= M \otimes \mathcal{B}(\ell^2(\mathbb{N})) & \widetilde{\mathcal{M}} &= \mathcal{M} \otimes \mathcal{B}(\ell^2(\mathbb{N})) = M \otimes \mathcal{B}(\ell^2(\mathbb{N})) \otimes \mathcal{B}(\ell^2(\mathbb{N})) \end{aligned}$$

*Step 4.1: there is a partial isometry $w \in \mathcal{M}$ with $ww^* \in \mathcal{Z}(\mathcal{A})$, $w^*w \in \mathcal{Z}(\mathcal{B})$ and such that*

$$w^* \mathcal{A} w = w^*w \mathcal{B}.$$

We find an infinite sequence $(v_n)_n$ of partial isometries in $\mathcal{A}$ such that $v_n v_n^* = p \otimes e_{1,1}$ for all $n$, and such that $\sum_n v_n v_n^*$ is the central support of $p \otimes e_{1,1}$ in $\mathcal{A}$. Similarly, we find an infinite sequence $(\tilde{v}_n)_n$ of partial isometries in $\mathcal{B}$ with $\tilde{v}_n \tilde{v}_n^* = q \otimes e_{1,1}$ for all $n$, and such that $\sum_n \tilde{v}_n^* \tilde{v}_n$ is a central projection in $\mathcal{B}$. The partial isometry $w = \sum_n v_n v \tilde{v}_n^*$ satisfies the conditions of step 4.1.

*Step 4.2: there is a unitary $u \in \widetilde{\mathcal{M}}$ such that*

$$u^* \widetilde{\mathcal{A}} u = \widetilde{\mathcal{B}}.$$

Because $\widetilde{\mathcal{A}}$ is a regular subalgebra of the factor $\widetilde{\mathcal{M}}$, we see that the normalizer of $\widetilde{\mathcal{A}}$ acts ergodically on $\mathcal{Z}(\widetilde{\mathcal{A}})$. We find an infinite sequence $(w_n)_n$ of partial isometries in $\widetilde{\mathcal{M}}$ with $w_n w_n^* = ww^*$ and $w_n^* w_n \in \mathcal{Z}(\widetilde{\mathcal{A}})$, such that $w_n^* \widetilde{\mathcal{A}} w_n = w_n^* w_n \widetilde{\mathcal{A}}$ and such that $\sum_n w_n^* w_n = 1$. Similarly, we find an infinite sequence $(\tilde{w}_n)_n$ of partial isometries in $\widetilde{\mathcal{M}}$ with $\tilde{w}_n \tilde{w}_n^* = w^*w$ and $\tilde{w}_n^* \tilde{w}_n \in \mathcal{Z}(\widetilde{\mathcal{B}})$, such that $\sum_n \tilde{w}_n^* \tilde{w}_n = 1$ and such that $\tilde{w}_n^* \widetilde{\mathcal{B}} \tilde{w}_n = \tilde{w}_n^* \tilde{w}_n \widetilde{\mathcal{B}}$. The unitary $u = \sum_n w_n w \tilde{w}_n^*$ satisfies $u \widetilde{\mathcal{A}} u^* = \widetilde{\mathcal{B}}$.

*Step 5: If $A$ and $B$ are properly infinite and $\mathcal{Z}(A)$ (respectively $\mathcal{Z}(B)$) does not admit a finite $\mathrm{Norm}_M(A)$-invariant (resp. $\mathrm{Norm}_M(B)$-invariant) trace, then $A$ and $B$ are unitarily conjugate.* Take a unitary $u \in \mathcal{B}(\ell^2(\mathbb{N})) \otimes \mathcal{B}(\ell^2(\mathbb{N})) \otimes M$ that conjugates $\ell^\infty(\mathbb{N}) \otimes \mathcal{B}(\ell^2(\mathbb{N})) \otimes A$ onto $\ell^\infty(\mathbb{N}) \otimes \mathcal{B}(\ell^2(\mathbb{N})) \otimes B$. The fact that $A$ is properly infinite means exactly that there is a partial isometry $w_1 \in \mathrm{M}_{1,\infty}(\mathbb{C}) \otimes A$ such that $w_1 w_1^* = 1$ and $w_1^* w_1 = 1 \otimes 1 \in \mathcal{B}(\ell^2(\mathbb{N})) \otimes A$. There is a similar isometry $\tilde{w}_2 \in \mathrm{M}_{1,\infty}(\mathbb{C}) \otimes B$. The unitary $u_1 = (1 \otimes \tilde{w}_1)u(1 \otimes w_1^*)$ in $\mathcal{B}(\ell^2(\mathbb{N})) \otimes M$ conjugates $\ell^\infty(\mathbb{N}) \otimes A$ onto $\ell^\infty(\mathbb{N}) \otimes B$.

Because $\mathcal{Z}(A)$ is abelian, we can identify it with the $\mathrm{L}^\infty(X)$ of some measure space $X$. We know that the ergodic action $\mathrm{Norm}_M(A) \curvearrowright X$ does not admit a finite invariant measure on $X$. So the associated orbit equivalence relation is of type $\mathrm{I}_\infty$, $\mathrm{II}_\infty$ or $\mathrm{III}$. In any case there is a non-singular isomorphism $\Delta : \mathbb{N} \times X \to X$ such that $\Delta(i, x)$ is in the same orbit as $x$, for all $i \in \mathbb{N}$ and almost all $x \in X$. This yields a unitary $w_2 \in \mathrm{M}_{1,\infty}(\mathbb{C}) \otimes M$ such that $w_2^* A w_2 = \ell^\infty(\mathbb{N}) \otimes A$. We find a similar unitary $\tilde{w}_2$ for $B$. Now that unitary $u_2 = \tilde{w}_2 u_1 w_2^* \in M$ conjugates $A$ onto $B$. $\qquad\square$

In both steps 4.1 and 4.2 of the proof of proposition 3.2, we have to pass to an amplification. The following example explains why we can not avoid this. Let $A_0 \subset M_0$ be any Cartan subalgebra of a $\mathrm{II}_1$ factor. Set

$$M = M_0 \otimes \mathrm{M}_2(\mathbb{C})$$
$$A = A_0 \otimes \mathrm{D}_2(\mathbb{C})$$
$$B = A_0 \otimes \mathrm{M}_2(\mathbb{C})$$

With $v = 1 \otimes e_{1,1}$, we clearly have that $vAv^* = vv^* Bvv^*$. Of course $v$ does not extend to a unitary that conjugates $A$ onto $B$ ($A$ and $B$ are not even isomorphic).

We really need the "abelian" amplification in step 4.1: $A \otimes \mathcal{B}(\ell^2(\mathbb{N}))$ and $B \otimes \mathcal{B}(\ell^2(\mathbb{N}))$ are not unitarily conjugate in $M \otimes \mathcal{B}(\ell^2(\mathbb{N}))$ because the smallest projection $p \in A \otimes \mathcal{B}(\ell^2(\mathbb{N}))$ that has full central support in $A \otimes \mathcal{B}(\ell^2(\mathbb{N}))$, has $\mathrm{Tr}(p) = 2$ while $q = 1 \otimes e_{1,1} \otimes e_{1,1} \in B \otimes \mathcal{B}(\ell^2(\mathbb{N}))$ has full central support in $B \otimes \mathcal{B}(\ell^2(\mathbb{N}))$, and $\mathrm{Tr}(q) = 1$.

Also the "factorial" amplification in step 4.2 is necessary: the von Neumann algebras $A \otimes \ell^\infty(\mathbb{N})$ and $B \otimes \ell^\infty(\mathbb{N})$ are not even isomorphic.

## 4. Every fundamental group of an equivalence relation is the fundamental group of a factor

Given a $\mathrm{II}_1$ equivalence relation $\mathcal{R}$, we construct a type $\mathrm{II}_1$ factor $M$ whose fundamental group is $\mathcal{F}(M) = \mathcal{F}(\mathcal{R})$. For a type $\mathrm{II}_1$ equivalence relation $\mathcal{R}$ on a probability space $(X, \mu)$, denote by $\mathcal{R}^\infty$ the infinite amplification on $X \times \mathbb{N}$. Given any type $\mathrm{II}_1$ factor $Q$, we can construct a new type $\mathrm{II}_\infty$ factor $\widetilde{M} = (Q \otimes \mathrm{L}^\infty(X \times \mathbb{N})) *_{\mathrm{L}^\infty(X \times \mathbb{N})} \mathrm{L}(\mathcal{R}^\infty)$. For convenience, we will denote $A = \mathrm{L}^\infty(X \times \mathbb{N})$ and $P = \mathrm{L}(\mathcal{R}^\infty)$.

It is now clear that $\mathcal{F}(\mathcal{R}) \subset \mathrm{mod}(\mathrm{Aut}(\widetilde{M})) = \mathcal{F}(p\widetilde{M}p)$ for any projection $p \in \widetilde{M}$ with finite trace. We show that, for the right choice of $Q$, we actually have $\mathcal{F}(\mathcal{R}) = \mathcal{F}(p\widetilde{M}p)$.

**Theorem 4.1.** *Let $\mathcal{R}$ be any type $\mathrm{II}_1$ equivalence relation on $(X, \mu)$. Denote by $P = \mathrm{L}(\mathcal{R}^\infty)$ the generalized group–measure space construction of the infinite amplification of $\mathcal{R}$. Denote by $A = \mathrm{L}^\infty(X \times \mathbb{N})$ the corresponding Cartan subalgebra $A \subset P$.*

*Let $Q$ be a type $\mathrm{II}_1$ factor with trivial fundamental group and such that there exists a diffuse Cartan subalgebra $Q_0 \subset Q$ with relative property (T). Define a type $\mathrm{II}_\infty$ factor $M = (Q \otimes$*

$A) *_A P$. *Then we have*

$$\mathrm{mod}(\mathrm{Aut}(M)) = \mathcal{F}(\mathcal{R}).$$

Popa showed in [P1] that the type $\mathrm{II}_1$ factor $Q = \mathrm{L}(\mathrm{SL}_2\mathbb{Z} \ltimes \mathbb{Z}^2)$ satisfies the conditions of theorem 4.1. In this case, the Cartan subalgebra $Q_0 = \mathrm{L}(\mathbb{Z}^2)$ has the relative property (T) in $Q$.

*Proof.* It is obvious that the fundamental group of $\mathcal{R}$ is included in $\mathrm{mod}(\mathrm{Aut}(M))$. For the other inclusion, it is sufficient to show that every automorphism of $M$ preserves $A$ up to a unitary in $M$.

The techniques to do this were developed by Ioana, peterson and Popa in [IPP]. In fact, we need a semifinite generalization of these results, but the original proofs generalize in a straightforward way. A complete proof for these generalizations is also given in my thesis [D]. For each result we use, we give both relevant references.

Let $\psi : M \to M$ be an automorphism of $M$, and fix a projection $p \in A$ with finite trace. Consider the word-length deformation $m_\rho : M \to M$ as in [IPP]. Because $Q_0 \subset Q$ has the relative property (T), we know that $m_\rho$ converges uniformly on the unit ball of $\psi(pQ_0)$.

We show that $\psi(pQ_0) \not\prec_M A$. If this were not the case, [V2, lemma 3.5] would yield a projection $q \in A$ with finite trace and such that

$$qQ \subset q(M \cap A') \underset{M}{\prec} \psi(p(M \cap Q_0')p).$$

But by [IPP, theorem 1.2.1] (generalized to the semifinite setting in [CH, theorem 2.4]), we know that the relative commutant of $pQ_0$ is in fact equal to $p(Q_0 \otimes A)$. Then the type $\mathrm{II}_1$ factor $qQ$ embeds into the abelian von Neumann algebra $\psi(p(Q_0 \otimes A))$. This contradiction shows that $\psi(pQ_0) \not\prec_M A$.

But all the unitaries of $\psi(pQ_0)$ commute with the abelian von Neumann algebra $\psi(pA)$, so [IPP, proposition 1.4.4 and theorem 4.3] (or rather [D, lemma 4.8 and theorem 4.6]) shows that $\psi(pA)$ embeds into $(Q \otimes A)$ or into $P$. Since $\psi(pA)$ is regular in $\psi(p)M\psi(p)$, it follows that $\psi(pA) \prec_M A$, see [IPP, theorem 1.2.1] (generalized to the semifinite setting in [CH, theorem 2.4]). Now, [V2, lemma 3.5] shows that $q(M \cap A') \prec \psi(M \cap A')$, for some projection $q \in A$.

We can apply the same argument to $\psi^{-1}$, and we obtain that also $\psi(r(M \cap A')) \prec_M M \cap A'$. Now proposition 3.2 shows that $\psi(M \cap A')$ is stably unitarily conjugate to $M \cap A'$. So we find a unitary $u \in \mathcal{B}(\ell^2(\mathbb{N})) \otimes \mathcal{B}(\ell^2(\mathbb{N})) \otimes M$ conjugating $\mathcal{B}(\ell^2(\mathbb{N})) \otimes \ell^\infty(\mathbb{N}) \otimes \psi(M \cap A')$ onto $\mathcal{B}(\ell^2(\mathbb{N})) \otimes \ell^\infty(\mathbb{N}) \otimes (M \cap A')$.

Observe that $M \cap A'$ is isomorphic to $Q^{*\infty} \otimes A$. By [IPP, theorem 6.3], we know that $*^\infty Q$ has trivial fundamental group. So we can assume that $u \in 1 \otimes \mathcal{B}(\ell^2(\mathbb{N})) \otimes M$. Since $\mathcal{R}^\infty$ is a type $\mathrm{II}_\infty$ equivalence relation, we can assume that $u \in 1 \otimes 1 \otimes M$, so $\psi(M \cap A')$ is unitarily conjugate to $M \cap A'$ inside $M$. The same is true for their respective centers, $\psi(A)$ and $A$. $\square$

## 5. The $\mathrm{II}_1$ factor case

We want to prove a result similar to theorem A, but for type $\mathrm{II}_1$ factors instead of equivalence relations. To do this, we combine theorem A with theorem 3.1.

Let $\Gamma \curvearrowright (X, \mu)$ be a free, ergodic, p.m.p. action, let $\pi : \Gamma \to \Lambda$ be a quotient morphism and let $\Lambda \curvearrowright (Y, \nu)$ be an ergodic, infinite measure preserving action. Define a new action $\Gamma \curvearrowright X \times Y$ by $g(x, y) = (gx, \pi(g)y)$. Denote the corresponding group measure space construction by $M = \mathrm{L}^\infty(X \times Y) \rtimes \Gamma$. Any $\Delta \in \mathrm{Centr}_{\mathrm{Aut}_{\mathrm{ns}}(Y)}(\Lambda)$ defines an automorphism $\psi_\Delta : M \to M$ by the formula $\psi_\Delta((a \otimes b)u_g) = (a \otimes \Delta_*(b))u_g$. Two different automorphisms like that are never unitarily conjugate. However, there is another natural class of automorphisms of $M$. Let $\omega : \Lambda \times Y \to \mathrm{S}^1$ be a 1-cocycle. Define unitaries $v_s \in \mathrm{L}^\infty(Y)$ by $v_s(y) = \omega(s, s^{-1}y)$. Then the formula $\varphi_\omega((a \otimes b)u_g) = (a \otimes bv_{\pi(g)})u_g$ defines an automorphism of $M$. Two such automorphisms $\varphi_\omega$ and $\varphi_{\tilde\omega}$ are unitarily conjugate if and only if $\omega$ and $\tilde\omega$ are cohomologous. Denote by $\mathrm{H}_1(\Lambda \curvearrowright Y)$ the group of all 1-cocycles $\omega : \Lambda \times Y \to \mathrm{S}^1$, identifying cohomologous cocycles.

The preceding paragraph shows that $\mathrm{Out}(M)$ contains at least the group

$$(2) \qquad \mathrm{Centr}_{\mathrm{Aut}_{\mathrm{ns}}(Y,\nu)}(\Lambda) \ltimes \mathrm{H}_1(\Lambda \curvearrowright Y) \subset \mathrm{Out}(M).$$

Of course, the $\varphi_\omega$ are trace preserving, so they do not contribute to $\mathrm{mod}(\mathrm{Aut}_{\mathrm{ns}}(M))$.

Theorem 5.1 shows that under strong conditions on $\Gamma \curvearrowright X$, the inclusion in (2) becomes and equality. Proposition 5.2 gives an explicit example of an action $\Gamma \curvearrowright (X, \mu)$ that satisfies the conditions of theorem 5.1. As with the equivalence relation case, the quotient group is $\Lambda = \mathbb{Q}^\times$. By [A, AN], this gives explicit examples of type $\mathrm{II}_1$ factors whose fundamental group can have any Hausdorff dimension $0 \le \alpha \le 1$.

**Theorem 5.1.** *Let $\Gamma \curvearrowright (X, \mu)$ be a free, ergodic and probability measure preserving action, and let $\pi : \Gamma \to \Lambda$ be a quotient morphism. Suppose that*

- *the conditions of theorem A are satisfied.*
- *the conditions of theorem C are satisfied.*
- $\ker \pi$ *is a perfect group.*

*Let $\Lambda \curvearrowright (Y, \nu)$ be an ergodic measure preserving action. Consider the action of $\Gamma$ on $X \times Y$ given by $g(x, y) = (gx, \pi(g)y)$, and denote the crossed product by $M = \mathrm{L}^\infty(X \times Y) \rtimes \Gamma$. Then the outer automorphism group of $M$ is*

$$\mathrm{Out}(M) = \mathrm{Centr}_{\mathrm{Aut}_{ns}(Y,\nu)}(\Lambda) \ltimes \mathrm{H}_1(\Lambda \curvearrowright Y).$$

*Proof.* By the discussion above, it is clear that the group in the right hand side is contained in the outer automorphism group of $M$.

Let $\psi : M \to M$ be an automorphism of $M$. By theorem 3.1, we can assume that $\psi$ globally preserves $\mathrm{L}^\infty(X \times Y)$. Theorem 5.1 in [S] yields an orbit equivalence $\Delta : X \times Y \to X \times Y$ such that $\psi(a \otimes b) = \Delta_*(a \otimes b)$ for all $a \otimes b \in A \otimes B$. By theorem A, we can assume that $\Delta$ is of the form $\mathrm{id} \times \Delta_0$ for some $\Delta_0 \in \mathrm{Centr}_{\mathrm{Aut}_{\mathrm{ns}}(Y)}(\Lambda)$.

In other words, we have that $\tilde{\psi} = \psi_{\Delta_0}^{-1} \circ \psi$ acts as the identity on $A \otimes B$. Theorem 3.1 in [S] yields a cocycle $\omega : \Gamma \times X \times Y \to \mathrm{S}^1$ such that $\tilde{\psi}$ can be described in the following way. Denote by $v_g \in \mathrm{L}^\infty(X \times Y)$ the unitary that is defined by $v_g(x,y) = \omega(g, g^{-1}x, \pi(g^{-1}y))$, for every $g \in \Gamma$. Then $\tilde{\psi}$ is given by $\tilde{\psi}((a \otimes b)u_g) = (a \otimes b)v_g u_g$.

Remember that $\ker \pi$ acts $\mathcal{U}_{\mathrm{fin}}$-cocycle superrigidly on $X$, so we can assume that $\omega(g, x, y)$ does not depend on the $x$-variable, at least whenever $g \in \ker \pi$. Since $\ker \pi$ acts weakly mixingly on $X$, [P4, proposition 3.6] tells us that $\omega(g, x, y)$ is independent of $x$ for all $g \in \Gamma$. From now on we consider $\omega : \Gamma \times Y \to \mathrm{S}^1$. For almost every $y \in Y$, the map $\ker \pi \ni g \mapsto \omega(g, y) \in \mathrm{S}^1$ is a group morphism. Since $\ker \pi$ is a assumed to be a perfect group, it follows that $\omega(g, y) = 1$ for all $g \in \ker \pi$ and almost all $y \in Y$. This shows that $\omega$ splits over the quotient $\pi$. We consider $\omega$ as a cocycle for the action $\Lambda \curvearrowright Y$. We have shown that $\tilde{\psi} = \varphi_\omega$.

For any automorphism $\psi : M \to M$, we have found a unitary $u \in M$, an automorphism $\Delta_0 \in \mathrm{Centr}_{\mathrm{Aut}_{\mathrm{ns}}(Y)}(\Lambda)$ and a cocycle $\omega \in \mathrm{H}_1(\Lambda \curvearrowright Y)$ such that $\psi$ is the composition $\mathrm{Ad}_u \circ \psi_{\Delta_0} \circ \varphi_\omega$. So we have shown that

$$\mathrm{Out}(M) = \mathrm{Centr}_{\mathrm{Aut}_{\mathrm{ns}}(Y)}(\Lambda) \ltimes \mathrm{H}_1(\Lambda \curvearrowright Y).$$

$\square$

**Proposition 5.2.** *Consider the group* $\Gamma = \mathrm{GL}_{n+1}\mathbb{Q} *_{\mathrm{T}_n\mathbb{Q} \ltimes \mathbb{Q}^n} (\mathrm{GL}_n\mathbb{Q} \ltimes \mathbb{Q}^n)$ *($n \geq 3$) where* $\mathrm{T}_n\mathbb{Q}$ *is the group of upper triangular matrices in* $\mathrm{GL}_n\mathbb{Q}$*. We view* $\mathrm{T}_n\mathbb{Q} \ltimes \mathbb{Q}^n$ *as the subgroup of* $\mathrm{GL}_{n+1}\mathbb{Q}$ *consisting of matrices of the form* $\left(\begin{smallmatrix} A & v \\ 0 & 1 \end{smallmatrix}\right)$ *with* $A \in \mathrm{T}_n\mathbb{Q}$ *and* $v \in \mathbb{Q}^n$*. Define a quotient map* $\pi : \Gamma \to \mathbb{Q}^\times$ *by* $\pi(A) = \det(A)$ *for all* $A \in \mathrm{GL}_{n+1}\mathbb{Q}$ *and* $\pi(A, v) = \det(A)$ *when* $(A, v) \in \mathrm{GL}_n\mathbb{Q} \ltimes \mathbb{Q}^n$*. Set* $H = \mathrm{GL}_n\mathbb{Q} \subset \mathrm{GL}_n\mathbb{Q} \ltimes \mathbb{Q}^n$ *and consider the generalized Bernoulli action* $\Gamma \curvearrowright (X, \mu) = (X_0, \mu_0)^{\Gamma/H}$ *over an atomic base space with unequal weights. For any ergodic action* $\mathbb{Q}^\times \curvearrowright (Y, \nu)$ *preserving the infinite non-atomic measure* $\nu$*. Define an action* $\Gamma \curvearrowright X \times Y$ *by the formula* $g(x, y) = (gx, \pi(g)y)$*, and denote the crossed product by* $M = \mathrm{L}^\infty(Y, \nu) \rtimes \Gamma$*. Then we have that*

$$\mathrm{Out}(M) = \mathrm{Centr}_{\mathrm{Aut}_{ns}(Y)}(\mathbb{Q}^\times) \ltimes \mathrm{H}_1(\mathbb{Q}^\times \curvearrowright Y).$$

*It follows that*

$$\mathrm{mod}(\mathrm{Aut}(M)) = \mathrm{mod}(\mathrm{Centr}_{\mathrm{Aut}_{ns}(Y)}(\mathbb{Q}^\times)).$$

*Proof.* The group $\Gamma$ clearly satisfies the conditions of theorem C with $G = \mathrm{SL}_n\mathbb{Z} \ltimes \mathbb{Z}^n$, so it suffices to check the conditions of theorem A. There are no non-trivial group morphisms $\theta : \mathrm{SL}_{n+1}\mathbb{Q} \to \mathbb{Q}^\times$ nor $\theta : \mathrm{SL}_n\mathbb{Q} \ltimes \mathbb{Q}^n \to \mathbb{Q}^\times$, so there are no non-trivial group morphisms to $\mathbb{Q}^\times$ from $\ker \pi = \mathrm{SL}_{n+1}\mathbb{Q} *_{\mathrm{ST}_n\mathbb{Q} \ltimes \mathbb{Q}^n} (\mathrm{SL}_n\mathbb{Q} \ltimes \mathbb{Q}^n)$. We denoted $\mathrm{ST}_n\mathbb{Q} = \mathrm{T}_n\mathbb{Q} \cap \mathrm{SL}_n\mathbb{Q}$.

Let $\omega : \ker \pi \times X \to \mathcal{G}$ be a cocycle to a $\mathcal{U}_{\mathrm{fin}}$ target group. Observe that $\mathrm{SL}_{n+1}\mathbb{Q}$ is a weakly rigid group, so by [P4, theorem 0.1] we can assume that $\omega|_{\mathrm{SL}_{n+1}\mathbb{Q}}$ is a group morphism. In particular the restriction of $\omega$ to $\mathbb{Q}^n$ does not depend on the $X$-variable. Proposition 3.6 in [P4] implies that $\omega|_{\mathrm{SL}_n\mathbb{Q} \ltimes \mathbb{Q}^n}$ is a group morphism. Because $\mathrm{SL}_{n+1}\mathbb{Q}$ and $\mathrm{SL}_n\mathbb{Q} \ltimes \mathbb{Q}^n$ generate $\ker \pi$, we have that $\omega$ itself is a group morphism.

It remains to prove the third condition. Let $\Delta$ be a self-conjugation of $\ker \pi \curvearrowright (X, \mu)$ with isomorphism $\delta : \ker \pi \to \ker \pi$. Observe that the rotations $A$ in $\mathrm{SL}_n\mathbb{Q}$ by $90°$, around

any $n-2$-dimensional linear subspace of $\mathbb{Q}^n$, can not be conjugated into $\mathrm{T}_n\mathbb{Q}\ltimes\mathbb{Q}^n$ inside $\mathrm{GL}_n\mathbb{Q}\ltimes\mathbb{Q}^n$. Hence they move every point $gH$ with $g\notin\mathrm{GL}_n\mathbb{Q}\ltimes\mathbb{Q}^n$, while the points $gH$ with $g\in\mathrm{GL}_n\mathbb{Q}\ltimes\mathbb{Q}^n$ form a copy of the affine space $\mathbb{Q}^n$. Therefore $H\cap\ker\pi$ acts with infinite orbits on $\Gamma/H-\{H\}$, so [V2, proposition 6.10] implies that $\Delta$ is of the form $\Delta(x)_{\alpha(i)}=x_i$ where $\alpha:\Gamma/H\to\Gamma/H$ is a conjugation with isomorphism $\delta$.

The isomorphism $\delta$ and the value $\alpha(H)$ completely determine $\alpha$. If $\alpha(H)=hH$, then $\delta(H\cap\ker\pi)=h(H\cap\ker\pi)h^{-1}$. Because $\Gamma_1=\mathrm{SL}_{n+1}\mathbb{Q}$ and $\Gamma_2=\mathrm{SL}_n\mathbb{Q}\ltimes\mathbb{Q}^n$ are weakly rigid groups, $\delta$ maps $\Gamma_1,\Gamma_2$ into conjugates of $\Gamma_i$ or $\Gamma_j$ with $i,j\in\{1,2\}$ (for an elementary proof, see for example point 4 of the proof of theorem 3.1 in [DV]). By symmetry, $\delta$ actually maps $\Gamma_1$ onto a conjugate $g_1\Gamma_1g_1^{-1}$ and $\Gamma_2$ onto $g_2\Gamma_2g_2^{-1}$. In particular, we find an automorphism $\delta_2=\mathrm{Ad}_{g_2^{-1}}\circ\delta|_{\Gamma_2}$ of $\Gamma_2$ that maps $\mathrm{SL}_n\mathbb{Q}$ onto a conjugate of itself. The same argument as in the proof of proposition 2.3 yields an $A\in\mathrm{GL}_n\mathbb{Q}\ltimes\mathbb{Q}^n$ such that $\delta_2=\mathrm{Ad}_A$. We can assume that $h=g_2A$. Now the isomorphism $\tilde\delta=\mathrm{Ad}_{A^{-1}g_2^{-1}}\circ\delta$ is identity on $\Gamma_2$, while it maps $\Gamma_1$ onto a conjugate $\tilde g_1\Gamma_1\tilde g_1^{-1}$. But the only elements of $\Gamma$ that conjugate $\mathrm{ST}_n\mathbb{Q}\ltimes\mathbb{Q}^n$ into $\Gamma_1$ are the elements of $\Gamma_1$, so we may assume that $\tilde g_1=e$. Now $\tilde\delta|_{\Gamma_1}$ is an isomorphism of $\mathrm{SL}_{n+1}\mathbb{Q}$ that is identity on $\mathrm{ST}_n\mathbb{Q}\ltimes\mathbb{Q}^n$. Such an isomorphism is easily seen to be the identity. We have proven that

$$\alpha(gH)=g_2AgH \quad\text{for all } gH\in\ker\pi/(H\cap\ker\pi)=\Gamma/H,$$

so we also have that $\Delta=g_2A\in\Gamma$. $\qquad\square$

## 6. A NON-ABELIAN GENERALIZATION

In this section we apply a kind of first quantization step to theorem D. We replace the measure space $(Y,\nu)$ by type $\mathrm{II}_\infty$ factor $B$, or more generally by a properly infinite von Neumann algebra $B$ with a semifinite trace Tr. Then we consider a trace preserving action $\beta$ of a group $\Lambda$ on $B$, such that $\Lambda$ acts ergodically on the center of $B$. As above, we take a free, ergodic, p.m.p. action $\alpha:\Gamma\curvearrowright(X,\mu)$ and a quotient $\pi:\Gamma\to\Lambda$. We define a new action $\sigma$ of $\Gamma$ on $\mathrm{L}^\infty(X,\mu)\overline{\otimes}B$ by the formula $\sigma_g(a\otimes b)=\alpha_g(a)\otimes\beta_{\pi(g)}(b)$. Consider the crossed product von Neumann algebra $M=(\mathrm{L}^\infty(X)\overline{\otimes}B)\rtimes\Gamma$.

Every automorphism $\psi_0$ of $B$ that commutes with the action of $\Lambda$ extends to an automorphism $\psi$ of $M$ by $\psi((a\otimes b)u_g)=(a\otimes\psi_0(b))u_g$. In fact, the requirement that $\psi_0$ really commutes with the action of $\Lambda$ is too strict. Let $\psi_0$ be an automorphism of $B$. We say that $\psi_0$ commutes with the action of $\Lambda$ up to a cocycle if there are unitaries $(v_s)_s$ in $B$ such that

$$\psi\circ\beta_s=\mathrm{Ad}_{v_s}\circ\beta_s\circ\psi \quad\text{for all } s\in\Lambda$$
$$\text{and}\quad v_{st}=v_s\beta_s(v_t) \quad\text{for all } s,t\in\Lambda.$$

If $\psi_0$ commutes with the action of $\Lambda$ up to the cocycle $(v_s)_s$, then we can extend $\psi_0$ to an automorphism $\psi$ of $M$ by the formula $\psi((a\otimes b)u_g)=(a\otimes\psi_0(b)v_{\pi(g)})u_g$.

Denote by $\mathrm{CCentr}_B(\Lambda)$ the group of all pairs $(\psi_0,(v_s)_s)$ where $\psi_0$ is an automorphism of $B$ that commutes with the action of $\Lambda$ up to the cocycle $(v_s)_s$. The group law in $\mathrm{CCentr}_B(\Lambda)$ is

defined by

$$(\psi_0, (v_s)_s) \cdot (\varphi_0, (w_s)_s) = (\psi_0 \circ \varphi_0, (\psi_0(w_s)v_s)_s).$$

We call this group the cocycle centralizer of the action of $\Lambda$ on $B$. Observe that the map $\theta : \mathrm{CCentr}_B(\Lambda) \to \mathrm{Out}(M)$ defined by

$$\theta(\psi_0, (v_s))((a \otimes b)u_g) = (a \otimes \psi_0(b)v_{\pi(g)})u_g$$

is a group morphism. If there is a unitary $w \in B$ such that $\psi_0 = \mathrm{Ad}_w$ and such that $v_s = w\beta_s(w^*)$ for all $s \in \Lambda$, then $\theta(\psi_0, (v_s))$ is trivial in $\mathrm{Out}(M)$. Such trivial elements $(\mathrm{Ad}_w, (w\beta_s(w^*))_s)$ will be called the inner elements of $\mathrm{CCentr}_B(\Lambda)$, and the group of all inner elements of $\mathrm{CCentr}_B(\Lambda)$ is denoted by $\mathrm{InnCCentr}_B(\Lambda)$. This is a normal subgroup of $\mathrm{CCentr}_B(\Lambda)$ and we denote the quotient by $\mathrm{OutCCentr}_B(\Lambda) = \mathrm{CCentr}_B(\Lambda)/\mathrm{InnCCentr}_B(\Lambda)$.

Under strong conditions on the action $\Gamma \curvearrowright (X, \mu)$, theorem 6.1 shows that the group morphism $\theta : \mathrm{OutCCentr}_B(\Lambda) \to \mathrm{Out}(M)$ defined above is an isomorphism.

**Theorem 6.1.** *Let $\pi : \Gamma \to \Lambda$ be a quotient morphism and let $\alpha : \Gamma \curvearrowright (X, \mu)$ be a free, ergodic, p.m.p. action such that the restriction to $\ker \pi$ is still ergodic. Let $\beta : \Lambda \curvearrowright (B, \mathrm{Tr})$ be a trace preserving action on a semifinite von Neumann algebra such that the restricted action of $\Lambda$ on $\mathcal{Z}(B)$ is ergodic. Define an action $\sigma : \Gamma \curvearrowright \mathrm{L}^\infty(X) \overline{\otimes} B$ by $\sigma_g(a \otimes b) = \alpha_g(a) \otimes \beta_{\pi(g)}(b)$. Consider the crossed product von Neumann algebra $M = (\mathrm{L}^\infty(X) \overline{\otimes} B) \rtimes \Gamma$.*

*Assume that*

- *the conditions of theorem 2.1 are satisfied.*
- *the conditions of theorem 5.1 are satisfied.*
- *$\ker \pi$ is a perfect group, i.e. there are no non-trivial group morphisms $\ker \pi \to \mathrm{S}^1$.*
- *$B$ is properly infinite and the action of $\Lambda \curvearrowright \mathcal{Z}(B)$ does not preserve any finite measure equivalent to the spectral measure.*

*Then the outer automorphism group of $M$ is*

$$\mathrm{Out}(M) = \mathrm{OutCCentr}_B(\Lambda).$$

*Every automorphism $\psi_0$ of $B$ that commutes with the action of $\Lambda$ up to a cocycle automatically scales the trace $\mathrm{Tr}$ by a constant $\mathrm{mod}(\psi_0)$. For any projection $p \in M$ with finite trace, the fundamental group of the type $\mathrm{II}_1$ factor $pMp$ is given by*

$$\mathcal{F}(pMp) = \mathrm{mod}(\mathrm{CCentr}_B(\Lambda)).$$

*Proof.* We show that the group morphism $\theta : \mathrm{OutCCentr}_B(\Lambda) \to \mathrm{Out}(M)$ defined above is in fact an isomorphism. We first show that $\theta$ is one-to-one. Suppose that $\theta(\psi_0, (v_s)_s)$ is inner in $M$. Then there is a unitary $u \in M$ such that

$$a = uau^* \text{ for all } a \in \mathrm{L}^\infty(X)$$

$$\psi_0(b) = ubu^* \text{ for all } b \in B$$

$$\text{and } v_{\pi(g)}u_g = uu_gu^* \text{ for all } g \in \Gamma.$$

The first equation implies that $u$ commutes with $\mathrm{L}^\infty(X)$, and since $\Gamma$ acts freely on $X$, it follows that $u \in A \otimes B$. Because $\ker \pi$ acts ergodically on $X$, the third equation shows that in fact $u \in B$. Now it is clear that $\psi_0 = \mathrm{Ad}_u$ and $v_s = u\beta_s(u^*)$ for all $s \in \Lambda$.

To prove the surjectivity of $\theta$, let $\psi : M \to M$ be an automorphism of $M$. Write $A = \mathrm{L}^\infty(X)$. By theorem 5.1, we find a unitary $u \in M$ such that $u\psi(A \otimes B)u^* = A \otimes B$. From now on, we assume that $\psi(A \otimes B) = A \otimes B$.

We can assume that $\psi(A \otimes B) = A \otimes B$.
*Step 1: there is a unitary $u \in M$ such that $u\psi(a)u^* = a$ for all $a \in A$.*
Because $\psi(A \otimes B) = A \otimes B$, we also have that $\psi(A \otimes \mathcal{Z}(B)) = A \otimes \mathcal{Z}(B)$. Identify $\mathcal{Z}(B)$ with $\mathrm{L}^\infty(Y, \nu)$ for some measure space $(Y, \nu)$. The action of $\Lambda$ on $B$ defines a non-singular action of $\Lambda$ on $Y$. The isomorphism $\psi : \mathrm{L}^\infty(X \times Y) \to \mathrm{L}^\infty(X \times Y)$ is of the form $\psi(f) = f \circ \Delta^{-1}$ for some orbit equivalence $\Delta : X \times Y \to X \times Y$ ([S]).

By theorem 2.1, we find an element $\varphi$ in the full group of $\mathcal{R}(\Gamma \curvearrowright X \times Y)$ and an element $\Delta_0 \in \mathrm{Centr}_{\mathrm{Aut}_{\mathrm{ns}}(Y)}(\Lambda)$ satisfying $\varphi(\Delta(x, y)) = (x, \Delta_0(y))$ almost everywhere. The unitary $u = u_\varphi \in \mathrm{L}^\infty(X \times Y) \rtimes \Gamma \subset M$ corresponding to $\varphi$ normalizes $A \otimes B$ and satisfies the relation $u\psi(a)u^* = a$ for all $a \in A$.

From now on, we assume that $\psi(a) = a$ for all $a \in A$. The unitaries $\overline{v}_g = \psi(u_g)u_g^*$, for $g \in \Gamma$, commute with $A$. Since $\Gamma$ acts freely on $X$, it follows that $\overline{v}_g \in A \otimes B$ for all $g \in \Gamma$.

*Step 2: there is a unitary $u \in A \otimes B$ such that $u\psi(u_g)u^*u_g^* \in B$ for all $g \in \Gamma$.*
We consider $\overline{v}_g$ as a function $\overline{v}_g : X \to \mathcal{U}(B)$, and we define a cocycle $\omega : \ker \pi \times X \to \mathcal{U}(B)$ by the formula $\omega(g, x) = \overline{v}_g(gx)$. We want to apply $\mathcal{U}_{\mathrm{fin}}$-cocycle superrigidity to the cocycle $\omega$, but $\mathcal{U}(B)$ is not necessarily a $\mathcal{U}_{\mathrm{fin}}$ group.

Let $p \in B$ be a projection with full central support in $B$ and such that $pBp$ is a finite von Neumann algebra. We find partial isometries $(w_n)_n$ in $B$ with $w_n^*w_n = p$ and such that $\sum_n w_n w_n^* = 1$. Consider the projection $q = \psi(p)$ as a map $q : X \to B$. Consider the Tr-preserving faithful normal semifinite extended center-valued trace $T : B^+ \to \widehat{\mathcal{Z}(B)^+}$. For every $g \in \ker \pi$, we know that $p$ commutes with $u_g$, so $q$ commutes with $\psi(u_g)$. It follows that

$$q(gx) = (u_g^* q u_g)(x) = (u_g^* \psi(u_g)\, q\, \psi(u_g)^* u_g)(x) = \overline{v}_g(x)q(x)\overline{v}_g(x)^*$$

for all $g \in \ker \pi$.

Hence the map $x \mapsto T(q(x))$ is invariant under the action of $\ker \pi$ on $X$. So all the projections $q(x)$ are equivalent in $B$. We find a partial isometry $w \in A \otimes B$ with $ww^* = q$ and such that $q_1 = w^*w \in B$. Remember that $p$ is a finite projection with full central support, as a projection in $B$, or equivalently as a projection in $A \otimes B$. It follows that the same is true for $q_1$. Hence we find partial isometries $(\widetilde{w}_n)_n$ in $B$ with $\widetilde{w}_n^*\widetilde{w}_n = q_1$ and such that $\sum_n \widetilde{w}_n\widetilde{w}_n^* = 1$.

Consider the unitaries $\tilde{v}_g = w^*\psi(u_g)wu_g^* \in q_1(A \otimes B)q_1$, for all $g \in \ker \pi$. When we consider the $\tilde{v}_g$ as functions $\tilde{v}_g : X \to \mathcal{U}(q_1 B q_1)$, they define a cocycle $\omega : \ker \pi \times X \to \mathcal{U}(q_1 B q_1)$, by the formula $\omega(g, x) = \tilde{v}_g(gx)$. Because $\mathcal{U}(q_1 B q_1)$ is a $\mathcal{U}_{\mathrm{fin}}$ group, cocycle superrigidity yields a unitary $v \in \mathcal{U}(A \otimes q_1 B q_1)$ such that $v\tilde{v}_g\sigma_g(v)^* \in B$ for all $g \in \ker \pi$.

Define a unitary $u \in A \otimes B$ by

$$u = \sum_n \widetilde{w}_n v w^* \psi(w_n^*) \in A \otimes B.$$

For all $g \in \ker \pi$, we see that $v_g = u\psi(u_g)u^*u_g^*$ is contained in $B$. We show that this is in fact true for all $g \in \Gamma$. Set $q_n = \widetilde{w}_n \widetilde{w}_n^*$ and observe that $q_n$ commutes with $v_g$ for all $g \in \ker \pi$. For any $g \in \Gamma$, we know that $v_g = u\psi(u_g)u^*u_g^*$ is contained in $A \otimes B$, so we can consider $v_g$ as a function $v_g : X \to \mathcal{U}(B)$. This function satisfies

$$v_g(hx) = v_h v_g(x) v_{g^{-1}hg}^* \text{ for all } h \in \ker \pi.$$

Since the $v_h$ with $h \in \ker \pi$ commute with the $q_n$, we see that the same relation holds for $q_n v_g$. This element $q_n v_g$ is contained in the polish space $\mathrm{L}^2(B, \widetilde{\mathrm{Tr}})$ where $\widetilde{\mathrm{Tr}}$ is a faithful, normal, semifinite trace on $B$ such that $\widetilde{\mathrm{Tr}}(q_n)$ is finite. Weak mixing (see [PV4, lemma 5.4]) implies that $q_n v_g \in B$. This is true for all $n$, so $v_g \in B$ for every $g \in \Gamma$. We can assume that $v_g = \psi(u_g)u_g^* \in B$ for all $g \in \Gamma$.

*Step 3: $v_g$ only depends on $\pi(g)$ and there is an isomorphism $\psi_0 \in \mathrm{CCentr}_B(\Lambda)$ such that, for all $a \in A$, $b \in B$ and $g \in \Gamma$, we have that $\psi((a \otimes b)u_g) = (a \otimes \psi_0(b))v_g u_g$.*

For any $b \in B$ with $\mathrm{Tr}(b^*b) < \infty$, we see that $c = \psi(b)$ commutes with $A$ and so $c \in A \otimes B$. We can consider $c$ as a function $c : X \to \mathrm{L}^2(B, \mathrm{Tr})$. Observe that, since $b$ commutes with all $u_g$ for $g \in \ker \pi$, we have

$$c(gx) = v_g c(x) v_g^* \text{ almost everywhere and for all } g \in \ker \pi.$$

Weak mixing (see [PV4, lemma 5.4]) shows that $c$ is essentially constant, or still, that $c \in B$. So $\psi$ maps $B$ into $B$. By symmetry, it follows that $\psi(B) = B$. Write $\psi_0 = \psi|_B$. Then we see that

$$\psi((a \otimes b)u_g) = (a \otimes \psi_0(b))v_g u_g \text{ for all } a \in A, b \in B \text{ and } g \in \Gamma.$$

We still have to show that $v_g$ only depends on $\pi(g)$. Since $(v_g)_g$ is a cocycle, it suffices to show that $v_g = 1$ for all $g \in \ker \pi$. Observe that $v_g$ is in the center of $B$ whenever $g \in \ker \pi$, and the application $\ker \pi \ni g \mapsto v_g \in \mathcal{U}(\mathcal{Z}(B))$ is a group morphism. Since $\ker \pi$ is a perfect group, this morphism is trivial. We have shown that $v_g = 1$ for every $g \in \ker \pi$.   $\square$

## 7. A FLEXIBLE CLASS OF EXAMPLES

We have given one explicit example of an action $\Gamma \curvearrowright (X, \mu)$ and a quotient $\pi : \Gamma \to \Lambda$ that satisfy the conditions of theorem 5.1, see proposition 5.2. In this example, the quotient group $\Lambda$ was abelian. For the applications in section 8, we need more flexibility in the choice of $\Lambda$.

We construct a new class of examples as follows. Let $\Gamma_1$ be a countable group and $\Sigma \subset \Gamma_1$ a subgroup. For any countable group $\Lambda$, we can consider $\Gamma = \Gamma_1 *_\Sigma (\Sigma \times \Lambda)$, together with the obvious quotient morphism $\pi : \Gamma \to \Lambda$. Let $H \subset \Gamma$ be a subgroup. We consider the generalized Bernoulli action $\Gamma \curvearrowright (X, \mu) = (X_0, \mu_0)^{\Gamma/H}$, over an atomic base space $(X_0, \mu_0)$ with unequal weights. Among the conditions we have to check, the least standard one says that $\mathrm{Norm}_{\mathrm{Aut}_{\mathrm{mp}}(X,\mu)}(\ker \pi) = \Gamma$.

If we have that $\text{Stab}_{\ker \pi}\{i\}j$ is infinite for all $i \neq j \in \Gamma/H$, then we know that the group of conjugations of $\ker \pi \curvearrowright X$ is given by $\text{Norm}_{\text{Aut}_{\text{mp}}(X,\mu)}(\ker \pi) = \text{Norm}_{\text{Perm}(\Gamma/H)}(\ker \pi)$ (see [V2, proposition 6.10]).

Observe that $\ker \pi$ is the infinite amalgamated free product of copies of $\Gamma_1$, amalgamated over $\Sigma$. The copies of $\Gamma_1$ correspond to the conjugates $\lambda \Gamma_1 \lambda^{-1} \subset \Gamma$, for $\lambda \in \Lambda$.

If we would choose $H$ to be a subgroup of $\Gamma_1$, then any permutation $\eta : \Lambda \to \Lambda$ defines an isomorphism $\delta_\eta : \ker \pi \to \ker \pi$ by the formula $\delta_\eta(\lambda g \lambda^{-1}) = \eta(\lambda)g\eta(\lambda)^{-1}$. for all $g \in \Gamma$ and $\lambda \in \Lambda$. The formula $\alpha_\eta(g\lambda H) = \delta_\eta(g)\eta(\lambda)H$ defines a conjugation $\alpha_\eta \in \text{Norm}_{\text{Perm}(\Gamma/H)}(\ker \pi)$. The permutation group of $\Lambda$ is uncountable, so it is certainly strictly larger that $\Gamma$.

We choose $H$ in the following way. For every $\lambda \in \Lambda$, we choose a "sufficiently different" subgroup $H_\lambda \subset \Gamma_1$, and we define $H$ to be the subgroup of $\Gamma$ that is generated by the $\lambda H_\lambda \lambda^{-1}$. More precisely, we have the following theorem.

**Theorem 7.1.** *Let $\Lambda$ be any countable group. Let $\Gamma_1$ be a property (T) group. Assume that $\Gamma_1$ is a perfect group and that there are no non-trivial group morphisms $\theta : \Gamma_1 \to \Lambda$.*

*Let $\Sigma \subset \Gamma_1$ be an amenable subgroup such that there is a finite set $g_1, \ldots, g_n \in \Gamma_1$ for which $\bigcap_i g_i \Sigma g_i^{-1}$ is finite. Assume that the elements of $\Sigma$ are the only elements $g \in \Gamma_1$ for which $g\Sigma g^{-1} \cap \Sigma$ has finite index in $\Sigma$.*

*For every $\lambda \in \Lambda$, choose an infinite subgroup $H_\lambda \subset \Gamma_1$, subject to the following conditions.*
- *If $\theta : \Gamma_1 \to \Gamma_1$ is an automorphism that maps a finite index subgroup of $H_\lambda$ into $H_\mu$, then it follows that $\lambda = \mu$ and $\theta = \text{Ad}_h$ for some $h \in H_\lambda$.*
- *If $\theta$ is an automorphism of $\Gamma_1$, then the intersection $\theta(H_\lambda) \cap \Sigma$ is trivial, for all $\lambda \in \Lambda$.*

*Consider the group $\Gamma = \Gamma_1 *_\Sigma (\Sigma \times \Lambda)$ with its natural quotient $\pi : \Gamma \to \Lambda$. Define $H$ to be the subgroup of $\Gamma$ that is generated by the $\lambda H_\lambda \lambda^{-1}$. Consider the generalized Bernoulli action $\Gamma \curvearrowright (X, \mu) = (X_0, \mu_0)^{\Gamma/H}$ over an atomic base space $(X_0, \mu_0)$ with unequal weights. Then the action $\Gamma \curvearrowright (X, \mu)$ satisfies the conditions of theorem 5.1.*

*Proof.* The action $\Gamma \curvearrowright X$ was constructed so that it satisfies the conditions of theorem C. Because $\ker \pi$ is generated by copies of the perfect group $\Gamma_1$, it is itself a perfect group. By the same argument, there are no non-trivial group morphisms $\theta : \ker \pi \to \Lambda$. It remains to show that $\ker \pi$ acts cocycle superrigidly on $X$ and that $\text{Norm}_{\text{Aut}_{\text{mp}}(X)}(\ker \pi) = \Gamma$. To prove this second condition, it is sufficient to show two properties of the action $\ker \pi \curvearrowright \Gamma/H$. We show that $H$ acts with infinite orbits on $\Gamma/H - \{H\}$, and we show that $\text{Norm}_{\text{Perm}(\Gamma/H)}(\ker \pi) = \Gamma$.

First we prove a general property of the subgroups $H_\lambda$ and $\Sigma$.
*Claim: Let $\delta : \Gamma_1 \to \Gamma_1$ be a group automorphism and let $g$ be an element of $\Gamma$. Then we have the following.*
- *If $G \subset H_\lambda$ is a finite index subgroup such that $g\theta(G)g^{-1}$ is contained in $H$, then $\theta$ is of the form $\theta = \text{Ad}_h$ for some $h \in \Gamma_1$ with $gh\lambda^{-1} \in H$.*
- *$\Sigma \cap g^{-1}Hg$ has infinite index in $\Sigma$.*
- *$\Sigma \cap H_\lambda = \{e\}$ for all $\lambda \in \Lambda$.*

Suppose that $g \subset H_\lambda$ is a finite index subgroup such that $g\theta(G)g^{-1}$ is contained in $H$. Possibly replacing $\theta$ by $\mathrm{Ad}_{g_0} \circ \theta$ for some $g_0 \in \Gamma_1$, we can assume that $g$ is an element of minimal length in $Hg\Gamma_1$. So we can write $g = \lambda_0 g_1 \ldots g_n \lambda_n$ with $\lambda_0 \in \Lambda$, $\lambda_1, \ldots, \lambda_n \in \Lambda - \{e\}$, with $g_1, \ldots, g_n \in \Gamma_1 - \Sigma$ and such that $g_1 \in \Gamma 1 - H_{\lambda_0}\Sigma$. The assumptions of the theorem yield an element $k \in G$ with $\theta(k) \notin \Sigma$. It follows that the expression

$$\lambda_0 g_1 \ldots g_n \lambda_n \theta(k) \lambda_n^{-1} g_n^{-1} \ldots g_1^{-1} \lambda_0^{-1}$$

is a reduced expression for $g\theta(k)g^{-1} \in H$. Since $g_1$ was not contained in $H_{\lambda_0}\Sigma$, this is only possible if $n$ was 0. In that case, we have that $\theta(G) \subset H_{\lambda_0}$. It follows that $\lambda_0 = \lambda$ and $\theta = \mathrm{Ad}_h$ for some $h \in H_\lambda$. Hence $gh\lambda^{-1} = \lambda h\lambda^{-1} \in H$. In this process we replaced $g$ by an element $h_0 g g_0$ in $Hg\Gamma_1$ and $\theta$ by $\mathrm{Ad}_{g_0^{-1}}$. This does not affect the conclusion.

Suppose that $G$ was a finite index subgroup of $\Sigma$ such that $gGg^{-1} \subset H$. We can assume that $g$ has minimal length among the elements of $Hg\Lambda$. So we can write $g = \lambda_1 g_1 \ldots \lambda_n g_n$ with $\lambda_1 \in \Lambda$, $\lambda_2, \ldots, \lambda_n \in \Lambda - \{e\}$, with $g_1, \ldots, g_n \in \Gamma_1 - \Sigma$ and such that $g_1 \notin H_{\lambda_1}\Sigma$. By the conditions on $\Sigma$, we find $k \in G$ such that $g_n k g_n^{-1} \notin \Sigma$. So we see that the expression

$$\lambda_1 g_1 \ldots \lambda_n (g_n k g_n^{-1}) \lambda_n^{-1} \ldots g_1^{-1} \lambda_1$$

is a reduced expression for an element of $H$. This is impossible: if $n \neq 1$, then we know that $\lambda_1 g_1$ can never be the first letters of an element of $H$. If $n = 1$, then it would follow that $g_1 k g_1^{-1} \in H_{\lambda_1}$, which is impossible because $e \neq k \in \Sigma$. This finishes the proof of the claim.

*Step 1: The action* $\ker \pi \curvearrowright (X, \mu)$ *is cocycle superrigid.* Let $\omega : \ker \pi \times X \to \mathcal{G}$ be a cocycle with a $\mathcal{U}_{\mathrm{fin}}$ target group. For every $\lambda \in \Lambda$, we know that the restriction $\omega|_{\lambda \Gamma_1 \lambda^{-1}}$ is cohomologous to a group morphism, by Popa's cocycle superrigidity theorem [P4, theorem 0.1]. So there are maps $\varphi_\lambda : X \to \mathcal{G}$ and group morphisms $\theta_\lambda : \lambda \Gamma_1 \lambda^{-1} \to \mathcal{G}$ such that

$$\omega(g, x) = \varphi_\lambda(gx)^{-1} \theta_\lambda(g) \varphi_\lambda(x)$$

for all $\lambda \in \Lambda$ and $g \in \lambda \Gamma_1 \lambda^{-1}$, and for almost all $x \in X$. These descriptions must match for $g \in \Sigma$, so we find that

$$(\varphi_\lambda \varphi_\mu^{-1})(hx) = \theta_\lambda(h)(\varphi_\lambda \varphi_\mu)(x)\theta_\mu(h)^{-1}$$

for all $h \in \Sigma$, $\lambda, \mu \in \Lambda$ and for almost all $x \in X$. The second assertion in our claim shows that $\Sigma$ acts weakly mixingly on $X$, so it follows that $\varphi_\lambda \varphi_\mu^{-1}$ is essentially constant (see for example [PV4, lemma 5.4]). Hence we can assume that all the $\varphi_\lambda$ are the same. Because the $\lambda \Gamma_1 \lambda^{-1}$ generate $\ker \pi$, we see that $\omega$ is cohomologous to a group morphism.

*Step 2: The group* $H$ *acts with infinite orbits on* $\Gamma/H - \{H\}$. We have to show that $gHg^{-1} \cap H$ has infinite index in $H$ whenever $g \in \Gamma - H$. This follows immediately from the claim above.

*Step 3: We show that* $\mathrm{Norm}_{\mathrm{Perm}(\Gamma/H)}(\ker \pi) = \Gamma$. Let $\alpha : \Gamma/H \to \Gamma/H$ be a conjugation for the action of $\ker \pi$. Possibly replacing $\alpha$ by $i \mapsto g\alpha(i)$ for some $g \in \Gamma$, we can assume that $\alpha(H) = H$. Denote by $\delta : \ker \pi \to \ker \pi$ the group automorphism such that $\alpha(gi) = \delta(g)\alpha(i)$ for all $g \in \ker \pi$ and $i \in \Gamma/H$. Remark that $\Delta(H) = H$.

Fix $\lambda \in \Lambda$. Because $\lambda \Gamma_1 \lambda^{-1}$ has property (T), we find an element $g \in \Gamma$ such that $\delta(\lambda \Gamma_1 \lambda^{-1}) \subset g\Gamma_1 g^{-1}$ (see for example [DV, part 4 of the proof of theorem 3.1] for an elementary proof). By

symmetry we actually have that $\delta(\lambda\Gamma_1\lambda^{-1}) = g\Gamma_1 g^{-1}$. Define an automorphism $\theta : \Gamma_1 \to \Gamma_1$ by $\theta(k) = g^{-1}\delta(\lambda k\lambda^{-1})g$. Then it follows that $g\theta(H_\lambda)g^{-1} \subset H$. Our claim above shows that $\theta$ is of the form $\theta = \mathrm{Ad}_h$ and that $h_\lambda = gh\lambda^{-1} \in H$.

For every $\lambda \in \Lambda$, we have found an element $h_\lambda \in H$ such that $\delta(\lambda k\lambda^{-1}) = h_\lambda \lambda k\lambda^{-1}h_\lambda^{-1}$ for all $k \in \Gamma_1$. For $k \in \Sigma$, these different descriptions have to match, so $h_\mu^{-1}h_\lambda$ commutes with $\Sigma$ for all $\lambda, \mu \in \Lambda$. The only element in $H$ that commutes with $\Sigma$ is $e$, so we have found one element $h \in H$ such that $\delta(k) = hkh^{-1}$ for all $k \in \ker\pi$. Replacing $\alpha(i)$ by $h^{-1}\alpha(i)$, we can assume that $\alpha$ commutes with the action of $\ker\pi$. We show that then $\alpha = \mathrm{id}$. Let $kH$ be an element in $\Gamma/H$, then we know that $kHk^{-1}$ fixes $\alpha(kH)$. But step 2 shows that $kHk^{-1}$ fixes only the point $kH$ itself. It follows that $\alpha(kH) = kH$. This works for any $k \in \Gamma$, so $\alpha = \mathrm{id}$. In the course of this argument, we changed $\alpha(i)$ to $g^{-1}\alpha(i)$ for some $g \in \Gamma$, so we have shown that $\alpha(i) = gi$ for all $i \in \Gamma/H$.                                                            $\square$

**Construction 7.2.** *Let $R = \mathrm{F}_2[X]$ be the ring of polynomials over the field $\mathrm{F}_2$ of two elements and take a natural number $k \geq 1$. Consider the group $\Gamma_1 = \mathrm{SL}_{3k}R \ltimes R^3$.*

*Let $\Lambda$ be any countable group for which there are no non-trivial group morphisms $\theta : \Gamma_1 \to \Lambda$. We will define the groups $\Sigma$ and $H_\lambda$ with $\lambda \in \Lambda$. Consider one symbol $* \notin \Lambda$ representing $\Sigma$. Take a one-to-one map $(\Lambda \sqcup \{*\}) \times \{1, \ldots, k\} \ni (\lambda, i) \mapsto n_{\lambda,i} \in \mathbb{N}$ and consider the matrices*

$$h_\lambda = \begin{pmatrix} h_{\lambda,1} & & \\ & \ddots & \\ & & h_{\lambda,k} \end{pmatrix} \quad where \quad h_{\lambda,i} = \begin{pmatrix} 0 & X^{2n_{\lambda,i}+1} & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

*Define subgroups $H_\lambda \subset \Gamma_1$ by the relation $H_\lambda = \{g \in \Gamma_1 \mid g(h_\lambda, 0) = (h_\lambda, 0)g\}$. Set $\Sigma = H_*$.*

*As in theorem 7.1, consider $\Gamma = \Gamma_1 *_\Sigma (\Sigma \times \Lambda)$, with its natural quotient morphism $\pi : \Gamma \to \Lambda$. Let $H$ be the subgroup generated by the $\lambda H_\lambda \lambda^{-1}$. Consider the generalized Bernoulli action $\Gamma \curvearrowright (X, \mu) = (X_0, \mu)^{\Gamma/H}$ over an atomic base space $(X_0, \mu_0)$ with unequal weights.*

**Proposition 7.3.** *The action $\Gamma \curvearrowright (X, \mu)$ and quotient $\pi : \Gamma \to \Lambda$ as in construction 7.2, satisfy the conditions of theorem C.*

*Proof.* We check that $\Gamma_1$, $\Sigma$ and the $H_\lambda$ satisfy the conditions of theorem 7.1.

The group $\Gamma_1$ is a property (T) group, see for example [BdlHV, example 3.4.1]. This group $\Gamma_1$ is a perfect group because $\Gamma_1$ is generated by elements of the form $x(1 + ae_{i,j}, 0)x^{-1}$ with $x \in \Gamma_1$, $a \in R$ and $i \neq j$. The symbol $e_{i,j}$ denotes the matrix whose $i,j$-th component is 1 while all other components are 0. The element $x(1 + ae_{i,j}, 0)x^{-1}$ is the commutator of $x(1 + ae_{k,j})x^{-1}$ with $x(1 + e_{i,k})x^{-1}$ where $k \neq i, j$.

Consider the field $E = \mathrm{F}_2(X)$ of rational functions over $\mathrm{F}_2$, and denote its algebraic closure by $\overline{E}$. Fix $\lambda \in \Lambda \sqcup \{*\}$ and $1 \leq i \leq k$. Observe that the characteristic polynomial of $h_{\lambda,i}$ is equal to $Y^3 + X^{2n_{\lambda,i}+1}Y + 1$. Denote its three roots in $\overline{E}$ by $s_{\lambda,i,1}, s_{\lambda,i,2}$ and $s_{\lambda,i,3}$. Consider the extension fields $E_{\lambda,i,j} = E(s_{\lambda,i,j})$ and remark that lemma 7.4 shows that $E_{\lambda,i,j} \cap E_{\lambda',i',j'} = E$ whenever $(\lambda, i, j) \neq (\lambda', i', j')$.

Consider the ring automorphism $\psi : R \to R$ that is defined by $\psi(X) = X + 1$. Remark that the only outer automorphism of $\Gamma_1$ is induced by this ring automorphism $\psi$. We still denote the group automorphism by $\psi$, and we extend the ring automorphism $\psi : R \to R$ to $\overline{E}$. This extension is unique up to multiplication by an element of the Galois group of $\overline{E}$ over $E$. Again it follows from lemma 7.4 that $E_{\lambda,i,j} \cap \psi(E_{\lambda',i',j'}) = E$ for any $\lambda, \lambda' \in \Lambda \sqcup \{*\}$, $1 \leq i, i' \leq k$ and $1 \leq j, j' \leq 3$.

The eigenvector $\xi_{\lambda,i,j}$ of $h_{\lambda,i}$ corresponding to the eigenvalue $s_{\lambda,i,j}$ is contained in $E_{\lambda,i,j}^3$ but not in $E^3$. Denote by $H_{\lambda,i} \subset \mathrm{SL}_3 R$ the subgroup of all matrices that commute with $h_{\lambda,i}$. Observe that this is exactly the group of all matrices whose eigenvectors are $\xi_{\lambda,i,1}, \xi_{\lambda,i,2}$ and $\xi_{\lambda,i,3}$.

Let $g \in H_{\lambda,i}$ be a nontrivial element. The $g$ has three distinct eigenvalues because if the eigenvalues corresponding to $\xi_{\lambda,i,1}$ and $\xi_{\lambda,i,2}$ were equal, then this eigenvalue must be contained in $E = E_{\lambda,i,1} \cap E_{\lambda,i,2}$. And hence so must the third eigenvalue. Now all three eigenvalues are invertible elements of $R$. The only such element is 1, so $g$ would be 1. Suppose that $k \in \mathrm{M}_3(R)$ is a possibly non-invertible matrix such that $gk \in kH_{\lambda',i'}$ for some $\lambda'$ and $i'$. Then $k\xi_{\lambda',i',j}$ is an eigenvector of $g$ and is contained in $E_{\lambda',i',j}$. Hence we see that either $k\xi_{\lambda',i',j} = 0$ or it follows that $\lambda' = \lambda$, $i' = i$ and $\xi_{\lambda,i,j}$ is an eigenvector of $k$. Because this holds for all $j = 1, 2, 3$, we conclude that $k$ commutes with $h_{\lambda,i}$. Similarly, if we have that $gk \in k\psi(H_{\lambda',i'})$, then it follows that $k = 0$.

We conclude that for every automorphism $\theta : \Gamma_1 \to \Gamma_1$ and for every $\lambda \neq \lambda' \in \Lambda \sqcup \{*\}$, the intersection $H_\lambda \cap \theta(H_{\lambda'})$ is trivial. Moreover, if $\theta$ maps a finite index subgroup of $H_\lambda$ into itself, then we have that $\theta = \mathrm{Ad}_h$ for some $h \in H_\lambda$.

Observe that $\Sigma$ is abelian, and hence amenable. Fix elements $g_i \in \Gamma_1 - H_{*,i}$ and consider the diagonal matrix $g = \mathrm{diag}(g_1, \ldots, g_k)$. Then it is clear that $g\Sigma g^{-1} \cap \Sigma$ is trivial and therefore finite. We have checked all conditions of theorem 7.1.  $\square$

**Lemma 7.4.** *Denote by $E = \mathrm{F}_2(X)$ the field of rational functions over $\mathrm{F}_2$. Let $a \in \mathrm{F}_2[X] \subset E$ be either $a = X^{2k+1}$ or $a = (X+1)^{2k+1}$ for some $k \in \mathbb{N}$. Consider the polynomial $P_a(Y) = Y^3 + aY + 1$, and denote the splitting field of $P_a$ over $E$ by $E_a$. Then we have that*

- *$E_a$ is a degree 6 extension of $E$*
- *$E_a \cap E_b = E$ whenever $b \neq a$ is of the form $X^{2l+1}$ or $(X+1)^{2l+1}$ for some $l \in \mathbb{N}$.*

*Proof.* Remark that $P_a(Y)$ is an irreducible polynomial for if $n, d \in \mathrm{F}_2[X]$ were coprime elements such that $P_a(n/d) = 0$, then it follows that $n = d = 1$, and hence that $a = 0$. So the degree $[E_a : E]$ is at least 3.

Denote the roots of $P_a(Y)$ in the algebraic closure of $E$ by $s_1, s_2, s_3 \in E_a$. Consider the element $t = s_1 s_2^2 + s_2 s_3^2 + s_3 s_1^2 \in E_a$. Observe that this element satisfies the second degree equation $Q_a(t) = t^2 + t + 1 + a^3 = 0$. Suppose $t$ were an element of $E$. Since $1 + a^3$ is a polynomial, the same is true for $t$. Then we know that $t^2 + t$ has even degree. This contradicts the fact that $1 + a^3$ has odd degree. So we see that $E \subsetneq E(t) \subsetneq E_a$, where $E(t)$ is a degree 2 extension. Hence $E_a$ is a degree 6 extension of $E$. Denote this intermediate field by $K_a = E(t)$.

Let $a \neq b$ be either $b = X^{2l+1}$ or $(X+1)^{2l+1}$. The intersection of fields $E_a \cap E_b$ is a normal subfield of $E_a$. So it is either $E_a$ itself, $K_a$ or $E$. We have to show that only the last case can occur. In both of the former cases, we see that $K_a = K_b$, because $K_a$ is the unique subfield of $E_a$ that has degree 2 over $E$. So we find elements $x, y \in E$ such that $xt + y$ satisfies $Q_b(xt + y) = 0$, while $Q_a(t) = 0$. This implies that $y^2 + y = a^3 + b^3$. Since the right hand side is a polynomial over $F_2$, the same is true for $y$. Hence $y^2 + y$ is a polynomial of even degree. If $k \neq l$, then we see that $a^3 + b^3$ is a polynomial of odd degree. So we can assume that $a = X^{2k+1}$ and $b = (X+1)^{2k+1}$. But now $X$ divides $y^2 + y$ but it does not divide $a^3 + b^3$. We have shown that $E_a \cap E_b = E$. $\qquad\square$

## 8. Applications

8.1. **Fundamental groups.** In section 5, we have shown the following. Let $\Lambda$ be a countable abelian group. Suppose $\Lambda$ acts ergodically and measure preservingly on an infinite measure space $(Y, \nu)$. Then there is a type $\mathrm{II}_1$ factor $M_\Lambda$ with fundamental group $\mathcal{F}(M_\Lambda) = \mathrm{mod}(\mathrm{Centr}_{\mathrm{Aut}_{ns}(Y)}(\Lambda))$. Using proposition 7.3, we can generalize this result.

With $\Lambda = \mathbb{F}_\infty$, proposition 7.3 yields and action $\Gamma \curvearrowright (X, \mu)$ and a quotient $\pi : \Gamma \to \mathbb{F}_\infty$ that satisfy the conditions of theorem C. Let $\mathbb{F}_\infty$ act ergodically (but not necessarily freely) on an infinite measure space $(Y, \nu)$. Then we find a type $\mathrm{II}_1$ factor $M$ with fundamental group

$$\mathcal{F}(M) = \mathrm{mod}(\mathrm{Centr}_{\mathrm{Aut}_{ns}(Y)}(\mathbb{F}_\infty)).$$

We give a slightly more appealing characterisation of the groups that appear in the right hand side. Let $(Y, \nu)$ be an infinite measure space and consider a closed subgroup $\mathcal{G}$ of measure preserving transformations on $Y$. Assume that $\mathcal{G}$ acts ergodically on $Y$. Since $\mathrm{Aut}_{mp}(Y, \nu)$ is a Polish group, so is $\mathcal{G}$ and hence we can take a group morphism $\rho : \mathbb{F}_\infty \to \mathcal{G}$ that has dense range in $\mathcal{G}$. This defines an ergodic measure preserving action of $\mathbb{F}_\infty$ with

$$\mathrm{Centr}_{\mathrm{Aut}_{ns}(Y)}(\mathbb{F}_\infty) = \mathrm{Centr}_{\mathrm{Aut}_{ns}(Y)}(\mathcal{G}).$$

We have proven the following theorem.

**Theorem 8.1.** *Let $(Y, \nu)$ be a standard infinite measure space. Let $\mathcal{G} \subset \mathrm{Aut}_{mp}(Y, \nu)$ be a closed subgroup of measure preserving transformations of $Y$. Assume $\mathcal{G}$ acts ergodically on $Y$. Then there is a type $\mathrm{II}_1$ equivalence relation $\mathcal{R}_\mathcal{G}$ and a type $\mathrm{II}_1$ factor $M_\mathcal{G}$ with fundamental group*

$$\mathcal{F}(M_\mathcal{G}) = \mathcal{F}(\mathcal{R}_\mathcal{G}) = \mathrm{mod}(\mathrm{Centr}_{\mathrm{Aut}_{ns}(Y)}(\mathcal{G})).$$

Using the results of section 6, we can further generalize theorem 8.1 to theorem 8.2 below. However, we can not give an explicit characterisation in full generality. The problem is the third condition of theorem 3.1, which requires that $\ker \pi$ contains a property (T) subgroup $G$ such that there are no $*$-homomorphisms $\theta : \mathrm{L}(G) \to pBp$ for any projection $p \in B$ with finite trace. For many choices of $B$, any property (T) group $G$ will suffice. For examples if $B$ is the hyperfinite $\mathrm{II}_1$ factor, or if $B = \mathcal{B}(\ell^2(\mathbb{N})) \otimes \mathrm{L}(\mathbb{F}_\infty)$. For an arbitrary $B$, we use Ozawa's result from [O2].

**Theorem 8.2** (see also theorem F). *Let $(B, \mathrm{Tr})$ be any separable properly infinite but semifinite von Neumann algebra, with a given semifinite trace* $\mathrm{Tr}$. *Let $\Lambda \subset \mathrm{Out}_{tp}(B, \mathrm{Tr})$ be a countable group of trace preserving automorphisms of $B$. Suppose that $\Lambda$ acts ergodically on the center of $B$. Then there is a type $\mathrm{II}_1$ factor $M_\Lambda$ with fundamental group*

$$\mathcal{F}(M_\Lambda) = \mathrm{mod}(\mathrm{Centr}_{\mathrm{Out}(B)}(\Lambda)),$$

*where $\mathrm{Centr}_{\mathrm{Out}(B)}(\Lambda)$ is the group of all outer automorphisms of $B$ that commute with $\Lambda$, as outer automorphisms.*

Conversely, for every separable $\mathrm{II}_1$ factor $M$, the fundamental group is of this form:

$$\mathcal{F}(M) = \mathrm{mod}(\mathrm{Aut}(\mathcal{B}(\ell^2(\mathbb{N})) \otimes M)) = \mathrm{mod}(\mathrm{Centr}_{\mathrm{Out}(\mathcal{B}(\ell^2(\mathbb{N})) \otimes M)}(\Lambda)).$$

*Proof.* Let $(B, \mathrm{Tr})$ and $\Lambda$ be as in the statement.
*Step 1: we can assume that there is no finite $\Lambda$-invariant measure on the center of $B$.*
Suppose there was such a finite measure. We replace $B$ by $B_1 = \ell^\infty(\mathbb{N}) \overline{\otimes} B$ and $\Lambda_0$ by $\Lambda_1 = \mathrm{Perm}_\infty \times \Lambda_0$ where $\mathrm{Perm}_\infty$ is the group of finite permutations of $\mathbb{N}$. The group morphism

$$\theta : \mathrm{Centr}_{\mathrm{Out}(B)}(\Lambda) \ni \psi \mapsto \mathrm{id} \otimes \psi \in \mathrm{Centr}_{\mathrm{Out}(B_1)}(\Lambda_1)$$

is an automorphism.

*Step 2: there is genuine action $\beta : \mathbb{F}_\infty \curvearrowright B$ such that*

$$\mathrm{mod}(\mathrm{CCentr}_B(\mathbb{F}_\infty)) = \mathrm{mod}(\mathrm{Centr}_{\mathrm{Out}(B)}(\Lambda)).$$

Take a quotient morphism $\rho : \mathbb{F}_\infty \to \Lambda \subset \mathrm{Out}(B)$. For every elementary generator $a_n$ of $\mathbb{F}_\infty$, choose a lift $\beta(a_n) \in \mathrm{Aut}(B)$ for $\rho(a_n) \in \mathrm{Out}(B)$. This choice extends to a group morphism $\beta : \mathbb{F}_\infty \to \mathrm{Aut}(B)$, that is necessarily a lift of the morphism $\rho : \mathbb{F}_\infty \to \mathrm{Out}(B)$. It is clear that $\mathrm{CCentr}_{\mathrm{Out}(B)}(\mathbb{F}_\infty) \subset \mathrm{Centr}_{\mathrm{Out}(B)}(\Lambda)$. Let $\psi$ be an automorphism of $B$ that commutes with $\Lambda$ up to inner automorphisms. For every elementary generator $a_n \in \mathbb{F}_\infty$, choose a unitary $v_{a_n} \in B$ such that $\psi \circ \beta_{a_n} = \mathrm{Ad}_{v_{a_n}} \circ \beta_{a_n} \circ \psi$. This choice extends uniquely to a cocycle $(v_s)_{s \in \mathbb{F}_\infty}$ such that $\psi$ commutes with $\mathbb{F}_\infty$ up to $(v_s)_s$. This shows that at least

$$\mathrm{mod}(\mathrm{CCentr}_B(\mathbb{F}_\infty)) = \mathrm{mod}(\mathrm{Centr}_{\mathrm{Out}(B)}(\Lambda)).$$

*Step 3: construction of $M_\Lambda$* By [O2, Theorem 2], we find a property (T) group $G$ such that there are no $*$-homomorphisms $\mathrm{L}(G) \to qBq$ for any projection $q \in B$ with finite trace. We can moreover assume that $G$ is ICC and quasifinite, meaning that all the proper subgroups of $G$ are finite. (see [O2, Theorem 1] which is a variant of [O1] and [G]) In particular, $G$ is a non-abelian simple group and therefore a perfect group.

Set $\widetilde{\Lambda} = G \times \mathbb{F}_\infty$ and construct a group $\Gamma$, a quotient $\tilde{\pi} : \Gamma \to \widetilde{\Lambda}$ and an action $\alpha : \Gamma \curvearrowright (X, \mu)$ as in construction 7.2. Consider the obvious quotient $\pi : \Gamma \to \mathbb{F}_\infty$. Define a new action $\sigma$ of $\Gamma$ on $\mathrm{L}^\infty(X) \overline{\otimes} B$ by the formula $\sigma_g(a \otimes b) = \alpha_g(a) \otimes \beta_{\pi(g)}(b)$. Denote the crossed product by $N = (\mathrm{L}^\infty(X) \overline{\otimes} B) \rtimes \Gamma$. Take any projection $p \in N$ with finite trace and set $M_\Lambda = pNp$.

*Step 3: the fundamental group of $M_\Lambda$ is*

$$\mathcal{F}(M_\Lambda) = \mathrm{mod}(\mathrm{Centr}_{\mathrm{Out}(B)}(\Lambda)).$$

It is sufficient to show that the action $\Gamma \curvearrowright X$ and the quotient $\pi : \Gamma \to \mathbb{F}_\infty$ from step 2, satisfy the conditions of theorem 6.1.

Proposition 7.3 shows that $\ker \tilde{\pi}$ acts weakly mixingly and cocycle superrigidly on $X$. By [P4, proposition 3.6] the same is true for $\ker \pi$. Since $\mathbb{F}_\infty$ has the Haagerup property, there are no non-trivial group morphisms $\theta : \ker \pi \to \mathbb{F}_\infty$. Proposition 7.3 shows that $\mathrm{Norm}_{\mathrm{Aut}_{\mathrm{mp}}(X)}(\ker \tilde{\pi}) = \Gamma$, but we have to show that $\mathrm{Norm}_{\mathrm{Aut}_{\mathrm{mp}}(X)}(\ker \pi) = \Gamma$. It suffices to show that every automorphism $\delta$ of $\ker \pi$ maps $\ker \tilde{\pi}$ onto itself. Remember that $\ker \pi = \ker \tilde{\pi} \times G$. Since $G$ is an infinite simple group, we know that there are no non-trivial group morphisms from $G$ to any finite group. Because $\Gamma_1 = \mathrm{SL}_3(\mathrm{F}_2[X]) \ltimes \mathrm{F}_2[X]^3$ is residually finite, there are no non-trivial group morphism $\theta : G \to \Gamma_1$. There can not be any non-trivial group morphisms $\theta : G \to \ker \tilde{\pi}$ because $G$ has property (T) and $\ker \tilde{\pi}$ is an (infinite) amalgamated free product of copies of $\Gamma_1$. So our automorphism $\delta : \ker \pi \to \ker \pi$ maps $G$ into $G$. By symmetry, $\delta$ maps $G$ isomorphically onto $G$. Since $G$ has trivial center, we see that $\delta$ maps $\ker \tilde{\pi} = \mathrm{Centr}_{\ker \pi}(G)$ onto itself.

The conditions of theorem 3.1 are satisfied by construction. Since $\ker \pi$ is generated by copies of the perfect groups $\Gamma_1$ and $G$, we see that $\ker \pi$ is a perfect group. Theorem 6.1 implies that the outer automorphism group of $M_\Lambda$ is given by

$$\mathrm{Out}(M_\Lambda) = \mathrm{OutCCentr}_B(\mathbb{F}_\infty).$$

By step 2, we see that

$$\mathcal{F}(M_\Lambda) = \mathrm{mod}(\mathrm{Centr}_{\mathrm{Out}(B)}(\Lambda)).$$

$\square$

8.2. **Outer automorphism groups.** Theorem A gives us a way to compute the outer automorphism group of type $\mathrm{II}_\infty$ equivalence relations $\mathcal{R}$ on an infinite measure space $(Y, \nu)$. For any set $U \subset Y$ with finite measure, we have the short exact sequence

$$1 \longrightarrow \mathrm{Out}(\mathcal{R}|_U) \longrightarrow \mathrm{Out}(\mathcal{R}) \xrightarrow{\mathrm{mod}} \mathcal{F}(\mathcal{R}|_U) \longrightarrow 1 \cdot$$

Once we know $\mathrm{Out}(\mathcal{R})$ and the group morphism mod, we can compute the outer automorphism group and the fundamental group of the type $\mathrm{II}_1$ equivalence relation $\mathcal{R}|_U$. A similar short exact sequence exists for the fundamental group and the outer automorphism group of type $\mathrm{II}_1$ factors.

Up to now, we have only computed fundamental groups of type $\mathrm{II}_1$ equivalence relations and factors. Theorems A and C allow us to also compute outer automorphism groups of type $\mathrm{II}_1$ equivalence relations and factors.

More concretely, let $\Lambda \curvearrowright (Y, \nu)$, $\Gamma \curvearrowright (X, \mu)$ and $\pi : \Gamma \to \Lambda$ be as in theorem A, and construct the $\mathrm{II}_\infty$ relation $\mathcal{R}$ as in this same theorem. Theorem A shows that the outer automorphism group of $\mathcal{R}$ is $\mathrm{Out}(\mathcal{R}) = \mathrm{Centr}_{\mathrm{Aut}_{\mathrm{ns}}(Y)}(\Lambda)$. But $\mathcal{R}$ is a type $\mathrm{II}_\infty$ equivalence relation. Take any subset $U \subset X \times Y$ with finite measure. Then the outer automorphism group of the type $\mathrm{II}_1$ equivalence relation $\mathcal{R}|_U$ is

$$\mathrm{Out}(\mathcal{R}|_U) = \mathrm{Centr}_{\mathrm{Aut}_{\mathrm{mp}}(Y,\nu)}(\Lambda),$$

where $\mathrm{Aut}_{\mathrm{mp}}(Y, \nu)$ is the set of all $\nu$-preserving automorphisms of $Y$.

The example from construction 7.2 satisfies the conditions of theorem A. This proves the following result.

**Theorem 8.3.** *For any closed subgroup $\mathcal{G} \subset \mathrm{Aut}_{mp}(Y, \nu)$ of $\nu$-preserving transformations that acts ergodically on a standard measure space (finite or infinite), there is a type $\mathrm{II}_1$ equivalence relation $\mathcal{R}_{\mathcal{G}}$ with outer automorphism group*

$$\mathrm{Out}(\mathcal{R}_{\mathcal{G}}) = \mathrm{Centr}_{\mathrm{Aut}_{mp}(Y,\nu)}(\mathcal{G}).$$

For any locally compact, second countable, unimodular group $G$, this yields a type $\mathrm{II}_1$ equivalence relation $\mathcal{R}$ with outer automorphism group $\mathrm{Out}(\mathcal{R}) = G$: let $G$ be such a locally compact second countable group unimodular group. Consider the measure space $(G, h)$ where $h$ is the Haar measure on $G$. By definition, the left translation action of $\mathcal{G} = G$ on $(G, h)$ is measure preserving. The centralizer of this action is precisely the right action of $G$ on $(G, h)$. A similar result was obtained in [PV2, theorem 1.1].

*Proof of theorem 8.3.* Let $\Lambda_0 \subset \mathcal{G}$ be a countable dense subgroup, and take a quotient morphism $\rho : \mathbb{F}_\infty \to \Lambda_0$. This gives us an ergodic measure preserving action $\mathbb{F}_\infty \curvearrowright (Y, \nu)$ with $\mathrm{Centr}_{\mathrm{Aut}_{mp}(Y,\nu)}(\mathbb{F}_\infty) = \mathrm{Centr}_{\mathrm{Aut}_{mp}(Y,\nu)}(\mathcal{G})$. Proposition 7.3 and theorem A show that there is a type $\mathrm{II}_1$ equivalence relation $\mathcal{R}_{\mathcal{G}}$ with

$$\mathrm{Out}(\mathcal{R}_{\mathcal{G}}) = \mathrm{Centr}_{\mathrm{Aut}_{mp}(Y,\nu)}(\mathcal{G}).$$

$\square$

Using exactly the same argument as above, we find a type $\mathrm{II}_1$ factor $M_{\mathcal{G}}$ with outer automorphism group

$$\mathrm{Out}(M_{\mathcal{G}}) = \mathrm{Centr}_{\mathrm{Aut}_{mp}(Y,\nu)}(\mathcal{G}) \ltimes \mathrm{H}_1(\mathbb{F}_\infty \curvearrowright Y),$$

where the action of $\mathbb{F}_\infty$ is given by a group morphism $\theta : \mathbb{F}_\infty \to \mathcal{G}$ with dense range. This result has not the same appeal as theorem 8.3 because the group $\mathrm{H}_1(\mathbb{F}_\infty \curvearrowright Y)$ is huge. In two special cases we can work around this problem. Theorem 8.4 below gives a type $\mathrm{II}_1$ factor $M$ with outer automorphism group $\mathrm{Out}(M) = \mathrm{SL}_n^{\pm}\mathbb{R} = \{g \in \mathrm{GL}_n\mathbb{R} \mid \det(g) = \pm 1\}$.

Besides that, theorem 8.5 shows that for every closed subgroup $\mathcal{G}$ of *probability* measure preserving transformations on $(Y, \nu)$, we find a $\mathrm{II}_1$ factor $M_{\mathcal{G}}$ with outer automorphism group $\mathrm{Out}(M_{\mathcal{G}}) = \mathrm{Centr}_{\mathrm{Aut}_{mp}(Y,\nu)}(\mathcal{G})$. For any compact group $G$, this yields an explicit construction of a type $\mathrm{II}_1$ factor with outer automorphism group equal to $G$. The existence of such $\mathrm{II}_1$ factors was already proven in [FV]. But the class of groups of the form $\mathrm{Centr}_{\mathrm{Aut}_{mp}(Y,\nu)}(\mathcal{G})$ contains more than just the compact groups. For example, we can take $\mathcal{G} = \{\mathrm{id}\}$. Theorem 8.5 shows that there is a type $\mathrm{II}_1$ factor whose outer automorphism group is precisely $\mathrm{Aut}_{mp}(Y, \nu)$, the group of probability measure preserving transformations on a standard probability space.

**Theorem 8.4.** *For every natural number $0 \neq n \in \mathbb{N}$ there is a type $\mathrm{II}_1$ factor $M$ whose outer automorphism group is*

$$\mathrm{Out}(M) = \mathrm{SL}_n^{\pm}\mathbb{R}.$$

*Proof.* Take $m = 4n + 1$, and consider the action of $\Lambda = \mathrm{SL}_m\mathbb{Z}$ on $Y = \mathrm{M}_{m,n}(\mathbb{R})$ by left multiplication. This action clearly preserves the Lebesgue measure $\nu$ on $Y$, and its centralizer is

$$\mathrm{Centr}_{\mathrm{Aut}_{\mathrm{mp}}(Y,\nu)}(\Lambda) = \mathrm{SL}_n^{\pm}\mathbb{R}.$$

By [PV4, theorem 1.3], we know that the action $\Lambda \curvearrowright Y$ is $\mathcal{U}_{\mathrm{fin}}$ cocycle superrigid. Since $\mathrm{SL}_m\mathbb{Z}$ is a perfect group, it follows that $\mathrm{H}_1(\Lambda \curvearrowright Y) = \{1\}$.

Choose any $k \geq m + 1$ that is also a multiple of 3. Using construction 7.2, theorem A and theorem C, we construct a type $\mathrm{II}_1$ factor $M$ with outer automorphism group

$$\mathrm{Out}(M) = \mathrm{SL}_n^{\pm}\mathbb{R}.$$

$\square$

**Theorem 8.5.** *Let $(Y,\nu)$ be a probability space, and let $\mathcal{G} \subset \mathrm{Aut}_{\mathrm{mp}}(Y,\nu)$ be a closed subgroup of measure preserving transformations. Then there is a type $\mathrm{II}_1$ factor $M_{\mathcal{G}}$ with outer automorphism group $\mathrm{Out}(M_{\mathcal{G}}) = \mathrm{Centr}_{\mathrm{Aut}_{\mathrm{mp}}(Y,\nu)}(\mathcal{G})$.*

*Proof.* This proof of theorem 8.5 uses a generalization of the co-induced action. The construction is explained in definition 8.7 below and some properties are given in lemma 8.8 below.

Let $\rho : \mathbb{F}_\infty \to \mathcal{G}$ be any group morphism with dense range. Denote by $a_n$ the $n$-th elementary generator of $\mathbb{F}_\infty$. Consider the group

$$(3) \qquad \Lambda = \underset{G}{\underbrace{\mathrm{SL}_3\mathbb{Z}}} \ltimes (\underset{A}{\underbrace{\mathbb{Z}^3}} \times ((\underset{B}{\underbrace{\mathbb{Z}^3 \times \mathbb{Z}^3}}) * \underset{F_1}{\underbrace{\mathbb{Z}^3}} * \underset{F_2}{\underbrace{\mathbb{Z}^3}} * \ldots)),$$

where $\mathrm{SL}_3\mathbb{Z}$ acts in the obvious way on each of the copies of $\mathbb{Z}^3$. Because each copy of $\mathbb{Z}^3$ plays a slightly different role in the following, we give them different names, as indicated in (3). Consider the obvious action of $\mathrm{SL}_3\mathbb{Z} \ltimes (\mathbb{Z}^3 \times \mathbb{Z}^3 \times \mathbb{Z}^3) = G \ltimes (A \times B)$ on $I = \mathbb{Z}^3 \times \mathbb{Z}^3 \times \mathbb{Z}^3$. This action extends to an action of $\Lambda$ on $I$, where the $F_n$ act trivially.

We define a cocycle $\omega : \Lambda \times I \to \mathbb{F}_\infty$ by the following relations.

$$\omega(g, i) = e \quad \text{for all } g \in G \ltimes (A \times B) \text{ and all } i \in I$$

$$\omega(f, (i_1, i_2, i_3)) = a_n^{\det(f, i_2, i_3)} \quad \text{for all } f \in F_n \text{ and all } (i_1, i_2, i_3) \in I = \mathbb{Z}^3 \times \mathbb{Z}^3 \times \mathbb{Z}^3.$$

In the last formula, we denoted $\det(f, i_2, i_3) \in \mathbb{Z}$ for the determinant of the matrix whose columns are $f, i_2$ and $i_3$.

Consider the generalized co-induced action of $\mathbb{F}_\infty \curvearrowright (Y,\nu)$, associated with $\omega : \Lambda \times I \to \mathbb{F}_\infty$, as explained in definition 8.7. Denote the resulting action by $\Lambda \curvearrowright (Y_1, \nu_1)$. Lemma 8.8 shows that this action is ergodic and that

$$\mathrm{Centr}_{\mathrm{Aut}_{\mathrm{ns}}(Y_1)}(\Lambda) = \mathrm{Centr}_{\mathrm{Aut}_{\mathrm{ns}}(Y)}(\mathcal{G}).$$

Following construction 7.2, we construct a group $\Gamma$, an action $\Gamma \curvearrowright (X,\mu)$ and a quotient $\pi : \Gamma \to \Lambda$. Consider the crossed product $M = \mathrm{L}^\infty(X \times Y_1) \rtimes \Gamma$.

Observe that $\ker \pi$ is generated by copies of $\mathrm{SL}_3(\mathrm{F}_2[X]) \ltimes \mathrm{F}_2[X]^3$. The group $\mathrm{SL}_3(\mathrm{F}_2[X]) \ltimes \mathrm{F}_2[X]^3$ is generated by conjugations of the subgroup $\mathrm{SL}_3(\mathrm{F}_2)$, because $(1+P(X)e_{i,j}, Q(X)e_i) = [(1+P(X)e_{k,j}, Q(X)e_k), (1+e_{i,k}, 0)]$ whenever $i, j, k$ are different elements in $\{1, 2, 3\}$. Lemma 8.6 below shows that there are no non-trivial group morphisms $\theta : \mathrm{SL}_3(\mathrm{F}_2) \to \mathrm{SL}_3 \mathbb{Z}$. Hence there are none from $\ker \pi$ into $\mathrm{SL}_3 \mathbb{Z}$. Because $\ker \pi$ is a perfect group, there are no non-trivial group morphisms $\theta : \ker \pi \to \Lambda$.

Proposition 7.3 and theorem 5.1 show that

$$\mathrm{Out}(M) = \mathrm{Centr}_{\mathrm{Aut}_{\mathrm{mp}}(Y_1, \nu_1)}(\Lambda) \ltimes \mathrm{H}_1(\Lambda \curvearrowright Y_1).$$

We show that the cohomology group $\mathrm{H}_1(\Lambda \curvearrowright Y_1)$ is trivial. Let $c : \Lambda \times Y_1 \to \mathrm{S}^1$ be a cocycle. By Popa's cocycle superrigidity theorem [P4, theorem 0.1], we can assume that $c(g, y)$ does not depend on $y$ whenever $g \in A$. But $A \subset \Lambda$ is a normal subgroup that acts weakly mixingly on $Y_1$. Hence [P4, proposition 3.6] shows that $c(g, y)$ is independent of $y$ for any $g \in \Lambda$, or still $c : \Lambda \to \mathrm{S}^1$ is a group morphism. This group morphism is trivial because $\Lambda$ is a perfect group.

We have shown that $\mathrm{H}_1(\Lambda \curvearrowright Y_1)$ is trivial and hence that

$$\mathrm{Out}(M) = \mathrm{Centr}_{\mathrm{Aut}_{\mathrm{mp}}(Y_1, \nu_1)}.$$

$\square$

**Lemma 8.6.** *There are no non-trivial group morphisms $\theta : \mathrm{SL}_3(\mathrm{F}_2) \to \mathrm{SL}_3 \mathbb{Z}$.*

*Proof.* Suppose that $\theta : \mathrm{SL}_3(\mathrm{F}_2) \to \mathrm{SL}_3 \mathbb{Z}$ is a group morphism. We can consider this morphism as a non-trivial representation $\theta : \mathrm{SL}_3(\mathrm{F}_2) \to \mathrm{GL}_3 \mathbb{C}$. The character of this representation is defined as $\chi_\theta(g) = \mathrm{tr}(\theta(g))$ and hence $\chi_\theta$ takes values in $\mathbb{Z}$. According to the atlas of finite groups[CCN+], the character table of $\mathrm{SL}_3(\mathrm{F}_2)$ is the following:

| $\begin{pmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{pmatrix}$ | $\begin{pmatrix} 1&0&1 \\ 0&1&0 \\ 0&0&1 \end{pmatrix}$ | $\begin{pmatrix} 0&1&0 \\ 0&0&1 \\ 1&0&0 \end{pmatrix}$ | $\begin{pmatrix} 1&1&0 \\ 0&1&1 \\ 0&0&1 \end{pmatrix}$ | $\begin{pmatrix} 0&1&1 \\ 1&1&0 \\ 1&0&0 \end{pmatrix}$ | $\begin{pmatrix} 1&1&1 \\ 1&1&0 \\ 1&0&0 \end{pmatrix}$ |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 |
| 3 | -1 | 0 | 1 | $r_7 + r_7^2 + r_7^4$ | $r_7^3 + r_7^5 + r_7^6$ |
| 3 | -1 | 0 | 1 | $r_7^3 + r_7^5 + r_7^6$ | $r_7 + r_7^2 + r_7^4$ |
| 6 | 2 | 0 | 0 | -1 | -1 |
| 7 | -1 | 1 | -1 | 0 | 0 |
| 8 | 0 | -1 | 0 | 1 | 1 |

where $r_7$ denotes a primitive 7-th root of unity. From this table, we see that all non-trivial characters of dimension 3 take at least one non-integer value. So $\theta : \mathrm{SL}_3(\mathrm{F}_2) \to \mathrm{SL}_3 \mathbb{Z}$ is trivial. $\square$

Let $\Lambda_0 \subset \Lambda$ be an inclusion of groups, and suppose that $\Lambda_0$ acts probability measure preservingly on a space $(Y_0, \nu_0)$. One possible construction for the co-induced action of $\Lambda$ associated with $\Lambda_0 \curvearrowright (Y_0, \nu_0)$ is the following. Set $I = \Lambda / \Lambda_0$ and consider the space $(Y, \nu) = (Y_0, \nu_0)^I$. The group $\Lambda$ acts in a natural way on $I$. Choose representatives $g_i \in \Lambda$ for the cosets of $\Lambda_0$. Then we define a cocycle $\omega : \Lambda \times I \to \Lambda_0$ by the formula $g g_i = g_{gi} \omega(g, i)$. The co-induced

action of $\Lambda$ on $(Y, \nu)$ is given by $(gy)_i = \omega(g, g^{-1}i)y_{g^{-1}i}$. Up to conjugation, this action does not depend on the choice of representatives $g_i \in \Lambda$.

We generalize this construction.

**Definition 8.7.** *Let $\Lambda \curvearrowright I$ be an action of a countable group on a countable set. Let $\omega : \Lambda \times I \to \Lambda_0$ be a cocycle. Suppose that $\Lambda_0$ acts probability measure preservingly on $(Y_0, \nu_0)$. Define an action of $\Lambda$ on $(Y, \nu) = (Y_0, \nu_0)^I$ by the formula $(gy)_i = \omega(g, g^{-1}i)y_i$. This action is called the generalized co-induced action of $\Lambda_0 \curvearrowright (Y_0, \nu_0)$, with respect to $\omega$.*

**Lemma 8.8.** *Let $\Lambda \curvearrowright I$ be an action of a countable group on a countable set, and let $\omega : \Lambda \times I \to \Lambda_0$ be a cocycle. Suppose that $\Lambda_0 \curvearrowright (Y_0, \nu_0)$ is a probability measure preserving action. Consider the generalized co-induced action $\Lambda \curvearrowright (Y, \nu) = (Y_0, \nu_0)^I$.*

- *If all orbits of $\Lambda \curvearrowright I$ are infinite, then the generalized co-induced action $\Lambda \curvearrowright (Y, \nu)$ is weakly mixing.*
- *Suppose that $\Lambda \curvearrowright I$ and $\omega$ satisfy the following three conditions.*
  - *$\Lambda$ acts transitively on $I$.*
  - *There exists an $i \in I$ such that (or equivalently, for all $i \in I$) $\omega$ maps the set $\mathrm{Stab}\{i\} \times \{i\}$ surjectively onto $\Lambda_0$*
  - *There exists an $i \in I$ such that (or equivalently, for all $i \in I$) the subgroup $S_i = \{g \in \Lambda \mid gi = i \text{ and } \omega(g, i) = e\}$ acts with infinite orbits on $I \setminus \{i\}$.*
  *Then we have that*

$$\mathrm{Centr}_{\mathrm{Aut}_{mp}(Y,\nu)}(\Lambda) = \mathrm{Centr}_{\mathrm{Aut}_{mp}(Y_0,\nu_0)}(\Lambda_0).$$

*Proof.* The first point can be proven exactly as the analogous result for generalized Bernoulli actions, see for example [PV1, proposition 2.3].

For every $\Delta_0 \in \mathrm{Centr}_{\mathrm{Aut}_{\mathrm{mp}}(Y_0,\nu_0)}(\Lambda_0)$, we define $\Delta \in \mathrm{Centr}_{\mathrm{Aut}_{\mathrm{mp}}(Y,\nu)}(\Lambda)$ by the formula $\Delta(y)_i = \Delta_0(y_i)$. It remains to show that every automorphism $\Delta \in \mathrm{Centr}_{\mathrm{Aut}_{\mathrm{mp}}(Y,\nu)}(\Lambda)$ is of this form. Let $\Delta : Y \to Y$ be a measure preserving automorphism that commutes with the action of $\Lambda$. Exactly the same argument as in the proof of [V2, proposition 6.10] yields p.m.p. automorphisms $\Delta_i : Y_0 \to Y_0$ and a bijection $\alpha : I \to I$ such that that *Delta* is given by

(4)                                    $(\Delta(y))_i = \Delta_i(y_{\alpha^{-1}(i)}).$

It follows that $\alpha(gi) = g\alpha(i)$ for all $g \in \Lambda$ and $i \in I$. For fixed $i \in I$, we see that $\alpha(i)$ is fixed under the action of $S_i$. But $S_i$ acts with infinite orbits on $I \setminus \{i\}$, so $\alpha(i) = i$. This works for all $i \in I$ so we see that $\alpha = \mathrm{id}$.

From (4) we also see that $\Delta_{gi}(\omega(g, i)y) = \omega(g, i)\Delta_i(y)$ for almost all $y \in Y_0$, and for all $g \in \Lambda$, $i \in I$. Fix $i \in I$. Since $\omega(\mathrm{Stab}\{i\} \times \{i\}) = \Lambda_0$, we see that $\Delta_i$ commutes with $\Lambda_0$, or still, $\Delta_i \in \mathrm{Centr}_{\mathrm{Aut}_{\mathrm{mp}}(Y_0,\nu_0)}(\Lambda)$. Moreover, we see that $\Delta_{gi} = \Delta_i$ for all $g \in \Lambda$. Because $\Lambda$ acts transitively on $I$, it follows that $\Delta_j = \Delta_i$ for all $j \in I$. Writing $\Delta_0 = \Delta_i$, we have shown that $\Delta$ is given by $\Delta(y)_i = \Delta_0(y_i)$. $\square$

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